PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A
MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

Relaxation Methods Applied to Engineering Problems. VIIA. Biharmonic Analysis as Applied to the Flexure and Extension of Flat Elastic Plates

L. Fox and R. V. Southwell

Phil. Trans. R. Soc. Lond. A 1945 **239**, 419-460
doi: 10.1098/rsta.1945.0001

RELAXATION METHODS APPLIED TO ENGINEERING PROBLEMS

VII A*. BIHARMONIC ANALYSIS AS APPLIED TO THE FLEXURE AND EXTENSION OF FLAT ELASTIC PLATES

By L. FOX, B.A. AND R. V. SOUTHWELL, F.R.S.

(*Received* 3 *March* 1941)

By extension of technique described in earlier papers, biharmonic analysis and the solution of the equation $\nabla^4 w = W$ are brought within range of the relaxation method. Special attention is given to the problem of a flat elastic plate which is either bent or stretched (the second case being that to which photo-elastic methods are commonly applied). In all, four cases are presented, since the edge conditions may specify either tractions or displacements both in the flexural and in the extensional problem: one example of each is treated. Mixed boundary conditions (tractions specified at some points, displacements at others) are not considered in this paper.

It would seem that only slight modifications of method will be required to deal with aeolotropic plates (which present much greater difficulties in an orthodox analysis).

Introduction and summary

1. The third paper of this series (Christopherson and Southwell 1938) was concerned with the approximate solution of problems in which the wanted function w is governed by Poisson's equation in two variables, namely,

$$\nabla^2 w = Z, \tag{1}$$

where ∇^2 stands for the operator $\partial^2/\partial x^2 + \partial^2/\partial y^2$ and Z is a specified function of x and y, together with a boundary condition which fixes the value either of w or of its normal gradient $\partial w/\partial \nu$. In special cases Z is zero everywhere: then (1) reduces to Laplace's equation

$$\nabla^2 w = 0, \tag{2}$$

and its solution is a problem in 'plane-potential theory'.†

This paper deals similarly with the equation

$$\nabla^4 w = W, \tag{3}$$

* This paper is numbered VII A to preserve its sequence between VII and VIII of this series: it has no special connexion with VII.

† In particular, (2) is presented in the problem of conformal transformation. This was treated in Part V of this series (Gandy and Southwell 1940).

 [Issued 15 October 1941—Published 10 October 1945

including the 'biharmonic equation'

$$\nabla^4 w = 0 \tag{4}$$

as a special case. (∇^2 again stands for the operator $\partial^2/\partial x^2 + \partial^2/\partial y^2$, and W, like Z, is a specified function of x and y.) The main difficulty of the new problem lies in the fact that a *double* boundary condition is now imposed on w: when $\nabla^2 w$ is specified at the boundary, equation (3) or (4) can be integrated in two separate steps, each involving the solution of (1) or (2); but such separation is not possible when both w and $\partial w/\partial \nu$ have to take specified values at the boundary, and in some problems still less tractable boundary conditions are presented. Orthodox methods then seek a solution in the form of an infinite series, each term of which severally satisfies one but not both of the two conditions (Love 1928*a*, *b*). Though formally satisfactory, a solution of this kind requires much laborious computation before it can be utilized in engineering design, and in stress analysis (for example) calculation is usually discarded in favour of experimental methods depending on the photo-elastic effect, notwithstanding that much labour is entailed both in the making of the necessary measurements and in their subsequent reduction.

2. Approximate methods, not restricted to particular shapes of boundary, will have applications in many branches of mathematical physics. (For example, very slow motion in two dimensions of a viscous incompressible liquid entails a stream-function which is biharmonic and vanishes, together with its normal gradient, at a fixed solid boundary.) Here we consider two problems taken from the Theory of Elasticity, both relating to a flat plate of uniform thickness and concerned with the distortion of its 'middle surface' (§ 4).

In the first problem the middle surface is bent without extension, and an equation of the type of (3) relates its transverse displacement w with the specified intensity W of the transverse loading. In the second the middle surface is stretched but is not bent, and according to the nature of the stress and strain in directions normal to this surface we have a case either of 'plane stress' or of 'plane strain'; but in either case the stress can be expressed in terms of a 'stress-function' χ which satisfies either the biharmonic equation (4) or (when body forces as well as edge tractions are operative) an equation of type (3). Exact mathematical analogies will here be shown to hold between the flexural and extensional problems as relating to a boundary of given shape; so every solution of the first problem also provides a solution of the second, and on that understanding photo-elastic methods (for example) could be used to determine flexural deformations.*

In both problems the boundary (i.e. edge) conditions may specify the values either (A) of displacements or (B) of tractions. Case A of the extensional and case B of the flexural problem are mathematically analogous, also case A of the flexural and case B

* It is obvious that the analogies must exist, but we are not aware of any published paper in which they are precisely formulated.

of the extensional (§§ 8–11). Thus, by four physically distinctive cases, only two distinct problems in mathematics are presented. One appears to be much more difficult than the other when attacked by orthodox methods, and to have received relatively little attention.

'Mixed' boundary conditions can of course be imposed, i.e. displacements may be specified at some parts of the boundary and tractions at others. Problems of this kind present great difficulty whether attacked by orthodox or by relaxation methods, and every example will call for some special device. They will not receive further notice in this paper.

3. Our first section starts from the governing equations of the flexural and extensional problems, and for both reduces the boundary conditions to forms which show that mathematically (cf. § 2) the cases calling for separate consideration are not four but two. Section II explains the theoretical bases of the relaxation treatment, and Section III deals with practical details. Solutions are obtained to four examples, namely, a case of specified edge displacements and a case of specified edge tractions in both the flexural and the extensional problem. The first and third example are relatively simple, being wanted for the detailed illustration of methods; the other two are fairly representative of problems confronted in practical work. Section IV contains a brief discussion of results.

No attempt has been made to review the numerous papers, concerned with approximate solutions, which have appeared in recent years. It is unlikely, in present circumstances, that a review of this kind could be made complete; and until wider experience becomes available, any judgment regarding the merits of different methods is premature.

I. THEORY

The flexural problem for thin plates

4. In the approximate theory of flexure (Rayleigh 1926; Love 1927), the distortion of a plate is specified in terms of w, the transverse displacement of its middle surface (i.e. the surface which lies midway between, and parallel with, its two flat faces). Taking axes Ox, Oy in this surface, and Oz to form with Ox, Oy a right-handed system, we treat w as positive when directed along Oz, and we adopt the same convention in regard to the transverse loading Z (measured as intensity per unit area of middle surface).* On this understanding the condition of equilibrium is

$$Z = D\nabla^4 w, \tag{5}$$

where D (the 'flexural rigidity') $= \dfrac{2}{3}\dfrac{Eh^3}{1-\sigma^2}$,

h denotes the half-thickness of the plate, and

E and σ are the Young's modulus and Poisson's ratio of its material.

* Inertial and body forces (directed transversely) may contribute to Z.

At the edge (i.e. the boundary of the middle surface*) it might be expected (and was assumed by Poisson†) that three external actions can be specified for every point, namely, the line intensities of flexural moment G, of torsional moment H, and of transverse (shearing) force N. But (as was first shown by Kirchhoff) the order of equation (5) is such that only two conditions are admissible, and in the case where edge tractions are specified the accepted boundary conditions are (Love 1927, § 313; Rayleigh 1926, § 216)‡

$$\left.\begin{aligned} G &= -D\left[\nabla^2 w - (1-\sigma)\left(\frac{\partial^2 w}{\partial s^2} + \frac{1}{\rho}\frac{\partial w}{\partial \nu}\right)\right], \\ N - \frac{dH}{ds} &= -D\left[\frac{\partial}{\partial \nu}\nabla^2 w + (1-\sigma)\frac{\partial}{\partial s}\left(\frac{\partial^2 w}{\partial s \partial \nu} - \frac{1}{\rho}\frac{\partial w}{\partial s}\right)\right], \end{aligned}\right\} \tag{6}$$

ρ denoting the radius of curvature, and ν, s having the senses indicated by figure 1.

No question of a third boundary condition arises when edge displacements are specified, i.e. when the deflexion and slope of the middle surface have given values at the boundary. This case (w and $\partial w/\partial \nu$ specified at every point on the boundary) we shall term 'case A' of the flexural problem (cf. § 2), 'case B' being governed by the boundary conditions (6).

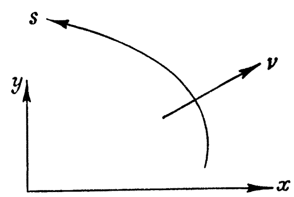


FIGURE 1

THE EXTENSIONAL PROBLEM FOR THIN PLATES

5. In the theory of plane strain (Coker and Filon 1931; Love 1927§), assuming that e_{zx}, e_{zy}, e_{zz} vanish severally and everywhere and that e_{xx}, e_{yy}, e_{xy} are independent of z, we satisfy the stress-equations of equilibrium, namely,

$$\frac{\partial X_x}{\partial x} + \frac{\partial X_y}{\partial y} + \rho X = 0, \quad \frac{\partial X_y}{\partial x} + \frac{\partial Y_y}{\partial y} + \rho Y = 0, \tag{i}$$

by writing

$$\left.\begin{aligned} X = -\frac{\partial \Omega_1}{\partial x}, \quad Y = -\frac{\partial \Omega_2}{\partial y}, \\ X_x = \frac{\partial^2 \chi}{\partial y^2} + \rho\Omega_1, \quad X_y = -\frac{\partial^2 \chi}{\partial x \partial y}, \quad Y_y = \frac{\partial^2 \chi}{\partial x^2} + \rho\Omega_2. \end{aligned}\right\} \tag{7}$$

Then the only 'strain-equation of compatibility' which is not satisfied identically, namely,

$$\frac{\partial^2 e_{xx}}{\partial y^2} + \frac{\partial^2 e_{yy}}{\partial x^2} = \frac{\partial^2 e_{xy}}{\partial x \partial y}, \tag{ii}$$

* Strictly defined, the 'boundary' is that closed curve in which the middle surface is cut by the (cylindrical) edge.

† Cf. Love (1927, § 297).

‡ Love's notation is adopted throughout this paper.

§ Love's treatment (§§ 144–6, 299–301) does not contemplate the operation of body forces.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

requires that $$\nabla^4\chi+\rho\nabla^2(\Omega_1+\Omega_2)-\frac{1}{1-\sigma}\rho\left(\frac{\partial^2\Omega_1}{\partial x^2}+\frac{\partial^2\Omega_2}{\partial y^2}\right)=0 \tag{8}$$

at all points within the boundary of the middle surface.

In (7) and (8) ρ denotes the density of the material and σ (as before) its Poisson's ratio, both quantities being uniform. If the body forces X, Y are conservative (so that $\partial X/\partial y=\partial Y/\partial x$), then Ω_1 and Ω_2 can be identified and (8) reduces to

$$\nabla^4\chi+\frac{1-2\sigma}{1-\sigma}\rho\nabla^2\Omega=0. \tag{8 A}$$

If in addition self-attraction of the material may be neglected (and this will almost always be the case), Ω is a plane-harmonic function of x and y, and χ accordingly satisfies the biharmonic equation

$$\nabla^4\chi=0. \tag{9}$$

The same expressions (7) and governing equation (8) hold in the theories of 'plane stress' and of 'generalized plane stress' except that in these σ is replaced by $\sigma'=\sigma(1+\sigma)$, therefore the factor $1/(1-\sigma)$, in (8), is replaced by $(1+\sigma)$ and the factor $(1-2\sigma)/(1-\sigma)$, in (8 A), is replaced by $(1-\sigma)$. Without restriction we may say that χ is always governed by an equation of the same form as (5) of § 4.

6. As in that section two main cases arise according as the boundary conditions specify displacements or tractions. In case A each of the component displacements u and v has a specified value at every point on the boundary, and instead of introducing χ it is more convenient to work throughout in terms of u and v, using the equations

$$\left.\begin{aligned}\frac{\partial\Delta}{\partial x}+A\left(\nabla^2u+\frac{\rho X}{\mu}\right)&=0,\\ \frac{\partial\Delta}{\partial y}+A\left(\nabla^2v+\frac{\rho Y}{\mu}\right)&=0,\end{aligned}\right\} \tag{10}$$

in which Δ stands for $\partial u/\partial x+\partial v/\partial y$, and (cf. § 5)

$$\left.\begin{aligned}A&=1-2\sigma,\text{ under conditions of plane strain,}\\ &=\frac{1-\sigma}{1+\sigma},\text{ under conditions of plane stress.}\end{aligned}\right\} \tag{11}$$

These are the equations of equilibrium expressed in terms of displacements (Love 1927, § 91).

7. In case B the edge tractions N_ν, N_s (figure 2) are specified, and we can deduce corresponding values of

$$\begin{aligned}N_\nu\cos(x,\nu)-N_s\sin(x,\nu)&\\ =X_\nu=X_x\cos(x,\nu)+X_y\sin(x,\nu)&\\ =\frac{\partial}{\partial s}\left(\frac{\partial\chi}{\partial y}\right)+\rho\Omega_1\cos(x,\nu),&\quad\text{according to (7),}\end{aligned}$$

and of $\quad N_\nu \sin(x,\nu) + N_s \cos(x,\nu)$

$$= Y_\nu = Y_y \sin(x,\nu) + X_y \cos(x,\nu)$$

$$= -\frac{\partial}{\partial s}\left(\frac{\partial \chi}{\partial x}\right) + \rho\Omega_2 \sin(x,\nu), \quad \text{according to (7).}$$

Accordingly $\partial\chi/\partial x$ and $\partial\chi/\partial y$ can be calculated for all points of the boundary from the expressions

$$\left.\begin{aligned} -\frac{\partial \chi}{\partial x} &= \int \{Y_\nu - \rho\Omega_2 \sin(x,\nu)\}\, ds, \\ \frac{\partial \chi}{\partial y} &= \int \{X_\nu - \rho\Omega_1 \cos(x,\nu)\}\, ds, \end{aligned}\right\} \tag{12}$$

in which the lower limits of integration are arbitrary. (Changes in these limits would entail additions to $\partial\chi/\partial x$, $\partial\chi/\partial y$ having the same values at every point on the boundary, therefore an addition to χ of the form $Ax+By$. This addition would not affect the stress-components.)

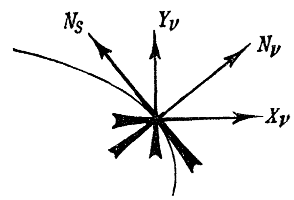


FIGURE 2

The boundary equations (12) are compatible with a single-valued expression for χ at all points of the boundary. For by Green's transformation

$$\oint \{X_\nu - \rho\Omega_1 \cos(x,\nu)\}\, ds = \oint X_\nu\, ds - \rho \iint \frac{\partial \Omega_1}{\partial x}\, dx\, dy$$

$$= \oint X_\nu\, ds + \rho \iint X\, dx\, dy, \quad \text{by (7)},$$

and this last expression, since it represents the resultant force in the x-direction of the forces acting on the whole plate, must vanish for equilibrium: therefore the expression (12) for $\partial\chi/\partial y$ is single-valued, and similarly the expression for $\partial\chi/\partial x$. Moreover the equation

$$\frac{\partial \chi}{\partial s} = \frac{dx}{ds}\cdot\frac{\partial \chi}{\partial x} + \frac{dy}{ds}\cdot\frac{\partial \chi}{\partial y}$$

can be integrated in the form

$$\chi = x\frac{\partial \chi}{\partial x} + y\frac{\partial \chi}{\partial y} - \int \left(x\frac{\partial^2 \chi}{\partial s\, \partial x} + y\frac{\partial^2 \chi}{\partial s\, \partial y}\right) ds \tag{13}$$

(again, with an arbitrary lower limit for the integral). We have seen that the first two terms in this expression are single-valued, and the same is true of the integral, since according to (12)

$$-\oint \left(x\frac{\partial^2 \chi}{\partial s\, \partial x} + y\frac{\partial^2 \chi}{\partial s\, \partial y}\right) ds = \oint [x\{Y_\nu - \rho\Omega_2 \sin(x,\nu)\} - y\{X_\nu - \rho\Omega_1 \cos(x,\nu)\}]\, ds$$

$$= \oint (xY_\nu - yX_\nu)\, ds + \rho \iint \left(y\frac{\partial \Omega_1}{\partial x} - x\frac{\partial \Omega_2}{\partial y}\right) dx\, dy$$

$$= \oint (xY_\nu - yX_\nu)\, ds + \rho \iint (xY - yX)\, dx\, dy.$$

This quantity measures the resultant moment about the z-axis of the external forces on the plate, and as such it must vanish for equilibrium.

Mathematical aspects

8. To summarize §§ 4–7, we have shown that

Case A of the flexural problem entails the satisfaction of (5) by a function w such that w and $\partial w/\partial \nu$, or (what is the same thing) $\partial w/\partial x$ and $\partial w/\partial y$, take specified values at the boundary;

Case B of the flexural problem entails the satisfaction of (5) by a function w such that the quantities G and $(N-\partial H/\partial s)$, as defined in (6), take specified values at the boundary;

Case A of the extensional problem entails the satisfaction of (10) by two functions u and v which take specified values at the boundary;

Case B of the extensional problem entails the satisfaction of (8) or (in special cases) of (8 A) or (9) by a function χ such that $\partial\chi/\partial x$ and $\partial\chi/\partial y$ take specified values at the boundary; these values being given by (12), in which X_ν, Y_ν, Ω_1 and Ω_2 are data of the problem.

Evidently, from a mathematical standpoint, case A of the flexural and case B of the extensional problem are identical. In both we have to solve an equation of the form

$$\nabla^4 w = W, \tag{3 bis}$$

W being specified, and to satisfy boundary conditions which fix the values of $\partial w/\partial x$, $\partial w/\partial y$. Nearly all of published investigations (by orthodox methods) relate to one or other of these cases.

We proceed to show that the two remaining cases (viz. case B of the flexural and case A of the extensional problem) can be similarly identified.

9. In the flexural problem, if

$$\left.\begin{aligned} &w = w_1 + w', \\ \text{where } \nabla^2 w_1 = 0 \text{ on the boundary, } &\nabla^4 w_1 = Z/D, \end{aligned}\right\} \tag{14}$$

then according to (5) $\nabla^2 w'$ is plane-harmonic, so that we may write

$$\nabla^2 w' = \frac{\partial^2 \phi}{\partial x \partial y}, \quad \text{where } \nabla^2 \phi = 0. \tag{15}$$

$$\left.\begin{aligned} &\text{Consequently} \quad \nabla^2 w = \nabla^2 w_1 + \frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial x \partial y} \quad \text{on the boundary,} \\ &\text{and} \quad \frac{\partial}{\partial \nu}\nabla^2 w = \frac{\partial}{\partial \nu}\left(\nabla^2 w_1 + \frac{\partial^2 \phi}{\partial x \partial y}\right) = \frac{\partial}{\partial \nu}\nabla^2 w_1 + \frac{\partial}{\partial s}\left(\frac{\partial^2 \phi}{\partial x^2}\right); \end{aligned}\right\} \tag{16}$$

so the boundary conditions (6) may be replaced by

$$\left.\begin{aligned} -\frac{G}{D} &= \frac{\partial^2 \phi}{\partial x \partial y} - (1-\sigma)\left(\frac{\partial^2 w}{\partial s^2} + \frac{1}{\rho}\frac{\partial w}{\partial \nu}\right), \\ \frac{1}{D}\left(\frac{\partial H}{\partial s} - N\right) - \frac{\partial}{\partial \nu}\nabla^2 w_1 &= \frac{\partial}{\partial s}\left\{\frac{\partial^2 \phi}{\partial x^2} + (1-\sigma)\left(\frac{\partial^2 w}{\partial s \partial \nu} - \frac{1}{\rho}\frac{\partial w}{\partial s}\right)\right\}. \end{aligned}\right\} \tag{17}$$

We integrate the second of (17) to obtain

$$\frac{1}{D}\left\{H-\int\left(N+D\frac{\partial}{\partial\nu}\nabla^2 w_1\right)ds\right\}=\frac{\partial^2\phi}{\partial x^2}+(1-\sigma)\left(\frac{\partial^2 w}{\partial s\,\partial\nu}-\frac{1}{\rho}\frac{\partial w}{\partial s}\right), \tag{18}$$

in which the quantity on the left is single-valued since

$$\left.\begin{aligned}\oint\left(N+D\frac{\partial}{\partial\nu}\nabla^2 w_1\right)ds &= \oint N\,ds+D\iint\nabla^4 w_1\,dx\,dy\\ &= \oint N\,ds+\iint Z\,dx\,dy,\quad \text{by (14)},\\ &= 0 \text{ for equilibrium of the whole plate.}\end{aligned}\right\} \tag{i}$$

Then from (18) and the first of (17) we have $\left(\text{remembering that } \frac{1}{\rho}=\frac{\partial}{\partial s}(x,\nu) \text{ and that } \frac{\partial^2\phi}{\partial x^2}=-\frac{\partial^2\phi}{\partial y^2}\right)$

$$\left.\begin{aligned}DA &= G\sin(x,\nu)-\cos(x,\nu)\left\{H-\int\left(N+D\frac{\partial}{\partial\nu}\nabla^2 w_1\right)ds\right\}\\ &= D\frac{\partial}{\partial s}\left[\frac{\partial\phi}{\partial y}-(1-\sigma)\left\{\cos(x,\nu)\frac{\partial w}{\partial\nu}-\sin(x,\nu)\frac{\partial w}{\partial s}\right\}\right]=D\frac{\partial}{\partial s}\left\{\frac{\partial\phi}{\partial y}-(1-\sigma)\frac{\partial w}{\partial x}\right\},\\ DB &= -G\cos(x,\nu)-\sin(x,\nu)\left\{H-\int\left(N+D\frac{\partial}{\partial\nu}\nabla^2 w_1\right)ds\right\}\\ &= D\frac{\partial}{\partial s}\left[\frac{\partial\phi}{\partial x}-(1-\sigma)\left\{\cos(x,\nu)\frac{\partial w}{\partial s}+\sin(x,\nu)\frac{\partial w}{\partial\nu}\right\}\right]=D\frac{\partial}{\partial s}\left\{\frac{\partial\phi}{\partial x}-(1-\sigma)\frac{\partial w}{\partial y}\right\},\end{aligned}\right\} \tag{19}$$

and we can integrate these expressions to obtain on the boundary

$$\left.\begin{aligned}\frac{\partial\phi}{\partial y}-(1-\sigma)\frac{\partial w}{\partial x} &= \int A\,ds=U\ (\text{say}),\\ \frac{\partial\phi}{\partial x}-(1-\sigma)\frac{\partial w}{\partial y} &= \int B\,ds=V\ (\text{say}).\end{aligned}\right\} \tag{20}$$

These boundary values of U and V are single-valued. For, since (cf. figure 1)

$$\cos(x,\nu)=\partial y/\partial s,\quad \sin(x,\nu)=-\partial x/\partial s,$$

we have in (20), from (19),

$$\left.\begin{aligned}\int A\,ds &= y\int\left(N+D\frac{\partial}{\partial\nu}\nabla^2 w_1\right)ds+\int\left\{G\sin(x,\nu)-H\cos(x,\nu)-y\left(N+D\frac{\partial}{\partial\nu}\nabla^2 w_1\right)\right\}ds,\\ \int B\,ds &= -x\int\left(N+D\frac{\partial}{\partial\nu}\nabla^2 w_1\right)ds-\int\left\{G\cos(x,\nu)+H\sin(x,\nu)-x\left(N+D\frac{\partial}{\partial\nu}\nabla^2 w_1\right)\right\}ds,\end{aligned}\right\} \tag{21}$$

and it has been shown in (i) above that the contour integral $\oint\left(N+D\frac{\partial}{\partial\nu}\nabla^2 w_1\right)ds$ must vanish for equilibrium of the plate as a whole. Moreover, since $\nabla^2 w_1$ vanishes on the boundary according to (14), we have by Green's theorem

$$D\oint x\frac{\partial}{\partial\nu}\nabla^2 w_1\,ds = D\oint\left[x\frac{\partial}{\partial\nu}-\frac{\partial x}{\partial\nu}\right]\nabla^2 w_1\,ds = D\iint x\nabla^4 w_1\,dx\,dy = \iint xZ\,dx\,dy,$$

$$D\oint y\frac{\partial}{\partial\nu}\nabla^2 w_1\,ds = D\oint\left[y\frac{\partial}{\partial\nu}-\frac{\partial y}{\partial\nu}\right]\nabla^2 w_1\,ds = D\iint y\nabla^4 w_1\,dx\,dy = \iint yZ\,dx\,dy.$$

Consequently, when the integrals in (21) are taken round the whole boundary, we obtain

$$-\oint A\,ds = \oint\{H\cos(x,\nu)-G\sin(x,\nu)+yN\}\,ds+\iint yZ\,dx\,dy,$$

$$-\oint B\,ds = \oint\{G\cos(x,\nu)+H\sin(x,\nu)-xN\}\,ds-\iint xZ\,dx\,dy,$$

and the quantities on the right of these equations are the resultant couples on the whole plate about axes Ox and Oy respectively, both of which must vanish for equilibrium.

10. Now let U and V be functions defined by (20) *not only on but also within the boundary*. Then, in virtue of (14) and (15), we have

$$\left.\begin{aligned}\nabla^2 U &= -(1-\sigma)\frac{\partial}{\partial x}\nabla^2(w_1+w'),\\ \nabla^2 V &= -(1-\sigma)\frac{\partial}{\partial y}\nabla^2(w_1+w'),\\ \text{and}\qquad \Delta &= (1+\sigma)\nabla^2 w'-(1-\sigma)\nabla^2 w_1,\end{aligned}\right\}\tag{22}$$

when Δ stands for

$$\frac{\partial U}{\partial x}+\frac{\partial V}{\partial y}.\tag{23}$$

Therefore

$$\left.\begin{aligned}\frac{\partial\Delta}{\partial x}+\frac{1+\sigma}{1-\sigma}\nabla^2 U+2X_1 &= 0,\\ \frac{\partial\Delta}{\partial y}+\frac{1+\sigma}{1-\sigma}\nabla^2 V+2Y_1 &= 0,\end{aligned}\right\}\tag{24}$$

where

$$X_1 = \frac{\partial}{\partial x}\nabla^2 w_1,\quad Y_1 = \frac{\partial}{\partial y}\nabla^2 w_1,\tag{25}$$

and $\nabla^2 w_1$ is defined by (14). So U and V, when their boundary values have been determined in accordance with (19) and (20), can be found from (22) exactly as in case A of the extensional problem (§ 6) u and v can be found from equations (10) which have the same mathematical form. Then, to complete the solution, we have only to deduce values of $\partial^2 w/\partial x^2$, $\partial^2 w/\partial y^2$, $\partial^2 w/\partial x\partial y$ from the expressions

$$\left.\begin{aligned}\frac{\partial^2 w}{\partial y^2}+\sigma\frac{\partial^2 w}{\partial x^2}-\nabla^2 w_1 &= \frac{\partial U}{\partial x},\\ \frac{\partial^2 w}{\partial x^2}+\sigma\frac{\partial^2 w}{\partial y^2}-\nabla^2 w_1 &= \frac{\partial V}{\partial y},\\ -2(1-\sigma)\frac{\partial^2 w}{\partial x\partial y} &= \frac{\partial U}{\partial y}+\frac{\partial V}{\partial x},\end{aligned}\right\}\tag{26}$$

which hold in virtue of (15) and (20). We can proceed (if it is required) to determine the deflexion w; but this last calculation will not be necessary if our concern is only with *stresses*, since according to (26)

$$-D\left(\frac{\partial V}{\partial y}+\nabla^2 w_1\right), \quad -D\left(\frac{\partial U}{\partial x}+\nabla^2 w_1\right), \quad -\tfrac{1}{2}D\left(\frac{\partial U}{\partial y}+\frac{\partial V}{\partial x}\right) \tag{27}$$

are expressions for the required stress-couples.

11. To summarize §§ 9–10, case B of the flexural problem (boundary values specified for G and for $N-\partial H/\partial s$) entails the finding (i) of a solution w_1 to equation (5) such that $\nabla^2 w_1$ vanishes on the boundary, and (ii) of solutions U and V to (22) which take specified values, calculable from (20) and (21), at the boundary. Part (i) of the problem can be solved by methods published previously, part (ii) is formally identical with case A of the extensional problem (§ 8).

Thus in both of these cases we have to solve simultaneous equations of the type of (10) in which A is a specified constant, X and Y are specified functions of x and y, and u and v take specified values on the boundary. This problem we shall term 'Mathematical Problem I', and the problem stated at the end of § 8 (comprising case A of the flexural and case B of the extensional problem) we shall term 'Mathematical Problem II'. 'Mixed' boundary conditions are not considered in this paper.

II. THEORETICAL BASES OF THE RELAXATION TECHNIQUE

12. Parts III and V of this series (Christopherson and Southwell 1938; Gandy and Southwell 1940) developed a technique of approximate solution for Laplace's and for Poisson's equation.* Assuming knowledge of that technique, we now describe extensions which bring biharmonic problems within its range.

'Non-dimensional' equations

To avoid unnecessary restriction, 'dimensional' quantities (i.e. quantities which depend upon the choice of units) should as far as possible be eliminated from the governing equations *before* these are subjected to numerical attack. In flexural problems, for example, we may write

$$x = Lx', \quad y = Ly', \quad w = \frac{Z_0 L^4}{D} w', \tag{28}$$

L denoting some governing dimension of the plate (e.g. the 'waist' of the specimen shown in figure 15), and Z_0 some specified intensity of loading (e.g. unit intensity in the units employed for L and D, or the actual intensity at the centre or some other reference point). Then equation (5), § 4, simplifies to

$$\left.\begin{aligned}&\nabla'^4 w = Z/Z_0 \text{ (a numerical function of } x' \text{ and } y'),\\ &\nabla'^2 \text{ denoting the operator } \partial^2/\partial x'^2+\partial^2/\partial y'^2.\end{aligned}\right\} \tag{3 A}$$

* The papers in question will hereafter be cited by the short titles 'Part III' and 'Part V'.

Its form is still given by (3), but now on the understanding that x, y, w, W are purely numerical; and any solution will apply to a whole family of plates, geometrically similar in plan-form and similarly loaded.

Similarly in extensional problems we may write

$$\left.\begin{aligned} x = Lx', \quad y = Ly', \quad u = Lu', \quad v = Lv',\\ \text{together with} \qquad X = \frac{\mu}{L\rho} X', \quad Y = \frac{\mu}{L\rho} Y'. \end{aligned}\right\} \tag{29}$$

Then equations (10) of § 6 simplify to

$$\left.\begin{aligned} &\frac{\partial \Delta'}{\partial x'} + A(\nabla'^2 u' + X') = 0,\\ &\frac{\partial \Delta'}{\partial y'} + A(\nabla'^2 v' + Y') = 0,\\ &\Delta' \text{ standing for } \frac{\partial u'}{\partial x'} + \frac{\partial v'}{\partial y'}. \end{aligned}\right\} \tag{10 A}$$

Their form is still given by (10), but now on the understanding that x, y, u, v, X, Y are purely numerical, and with ρ/μ replaced by unity. Here too, any solution will apply to a whole family of similar and similarly loaded plates.

Alternatives to (28) and (29) may be preferable in particular problems. These are merely examples of 'non-dimensional' treatment.

Finite-difference approximations

13. Part III based its approximate methods on the relation*

$$\frac{1}{N} \Sigma_{a,N}(w) - w_0 = \frac{a^2}{4} (\nabla^2 w)_0 + \frac{a^4}{64} (\nabla^4 w)_0 + \frac{a^6}{2^2 \cdot 4^2 \cdot 6^2} (\nabla^6 w)_0 + \ldots,$$

which is accurate as regards terms of order less than N in a. It showed (§ 8) that practicable values of N are 3, 4 and 6, and accordingly proposed the approximations

$$\left.\begin{aligned} \frac{1}{N} \Sigma_{a,N}(w) - w &= \frac{a^2}{4} \nabla^2 w, && \text{for use when } N = 3 \text{ or } 4,\\ &= \frac{a^2}{4} \nabla^2 w + \frac{a^4}{64} \nabla^4 w, && \text{for use when } N = 6, \end{aligned}\right\} \tag{30}$$

as accurate to the order of the terms retained. $\nabla^2 w$ being specified (as it is in problems governed by Poisson's equation) these relations impose values on the quantity

$$\left\{\frac{1}{N} \Sigma_{a,N}(w) - w\right\}$$

* Cf. equation (7), § 7, of the paper cited, in which Z stands for $\nabla^2 w$.

at every nodal point inside the boundary of the chosen net, so reducing the problem (from an approximate standpoint) to that of solving simultaneous equations equal in number to these nodal points. No attempt was made to solve the equations directly: instead, a standard 'relaxation pattern' was employed to 'liquidate' (i.e. to dispose by degrees of) the specified loading.

Here, in addition to such treatment of the operator ∇^2, we require a similar treatment of the several operators

$$\frac{\partial^2}{\partial x^2}, \quad \frac{\partial^2}{\partial y^2}, \quad \frac{\partial^2}{\partial x \partial y}, \quad \nabla^4 \equiv \left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right]^2,$$

which (with purely numerical significance for x and y) are presented in equations (3 A) and (10 A) of § 12. This problem we now examine.

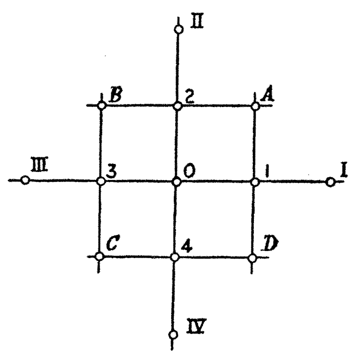


FIGURE 3

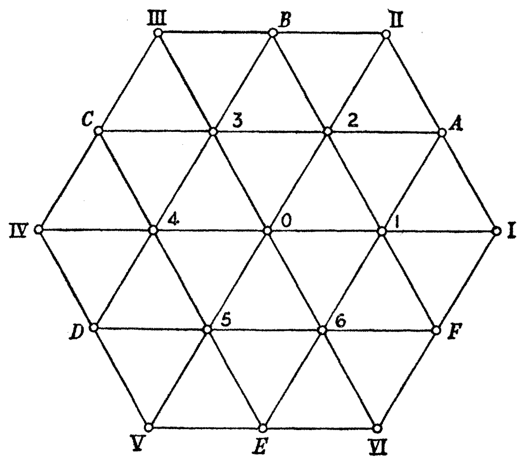


FIGURE 4

14. First, in regard to ∇^4 we have according to (30)

$$\left.\begin{aligned} \frac{1}{N}\Sigma_{a,N}(\nabla^2 w) - \nabla^2 w &= \frac{a^2}{4}\nabla^4 w, && \text{when } N = 3 \text{ or } 4, \\ &= \frac{a^2}{4}\nabla^4 w + \frac{a^4}{64}\nabla^6 w, && \text{when } N = 6, \end{aligned}\right\} \quad (31)$$

the first expression having an unknown error of order a^3 or a^4, the second an unknown error of order a^6.

Nets of hexagonal mesh ($N = 3$) offer no practical advantage, but square-mesh nets are convenient in relation to biharmonic problems. Here (figure 3) we have, according to the first of (31) with $N = 4$,

$$a^4(\nabla^4 w)_0 = (a^2\nabla^2 w)_1 + \ldots + (a^2\nabla^2 w)_4 - 4(a^2\nabla^2 w)_0, \quad \text{(i)}$$

and from the first of (30) with $N = 4$, five relations of which

$$(a^2\nabla^2 w)_0 = (w_1 + w_2 + w_3 + w_4) - 4w_0 \quad \text{(ii)}$$

is typical. Substituting from the latter in (i), we deduce that

$$a^4(\nabla^4 w)_0 = \Sigma_4(w_I) + 2\,\Sigma_4(w_A) - 8\,\Sigma_4(w_1) + 20w_0, \tag{32}$$

$\Sigma_4(w_I)$, $\Sigma_4(w_A)$, $\Sigma_4(w_1)$ standing for the sum of the w-values at the four symmetrical points typified by I, A, 1 respectively in figure 3.

For a triangular net (figure 4) we have similarly, from the second of (31) with $N = 6$,

$$\frac{a^4}{16}(\nabla^4 w) + \frac{a^6}{4.64}(\nabla^6 w) = \frac{1}{6}\left\{\left(\frac{a^2}{4}\nabla^2 w\right)_1 + \ldots + \left(\frac{a^2}{4}\nabla^2 w\right)_6\right\} - \left(\frac{a^2}{4}\nabla^2 w\right)_0, \tag{iii}$$

and from the second of (30) seven relations of which

$$\frac{a^2}{4}(\nabla^2 w)_0 = -\frac{a^4}{64}(\nabla^4 w)_0 + \tfrac{1}{6}(w_1 + w_2 + \ldots + w_6) - w_0 \tag{iv}$$

is typical. Substituting from the latter on the right-hand side of (iii), and using the approximation

$$\tfrac{1}{6}\Sigma_6(\nabla^4 w)_1 - (\nabla^4 w)_0 = \frac{a^2}{4}(\nabla^6 w)_0 + \ldots, \tag{v}$$

we find that

$$\frac{9}{4}\left[a^4(\nabla^4 w)_0 + \frac{a^6}{8}(\nabla^6 w)_0\right] = \Sigma_6(w_I) + 2\Sigma_6(w_A) - 10\Sigma_6(w_1) + 42w_0. \tag{33}$$

15. This derivation of (32) is not entirely convincing, because terms of order a^4 (and so comparable with its left-hand side) were disregarded in formulating the five relations of type (ii). But we can, working backwards, verify that the terms neglected in (32) are in fact of order a^6 at least.

In Part III, § 6, the summation $\Sigma_{a,N}(w)$ related to N points symmetrically grouped round a central point at a distance a, one point lying on the reference line $\theta = 0$. If, maintaining the symmetrical grouping, we rotate the points so that one lies on the line $\theta = \beta$, then a similar argument yields the relation

$$\frac{1}{N}\Sigma_{a,N}(w) = A_0(a) + A_N(a)\cos N\beta + A_{2N}(a)\cos 2N\beta + \ldots$$
$$+ B_N(a)\sin N\beta + B_{2N}(a)\sin 2N\beta + \ldots$$

—a generalization of (4) of the earlier paper. In particular, when $N\beta = \pi$ we have

$$\frac{1}{N}\Sigma_{a,N}(w) = A_0(a) - A_N(a) + A_{2N}(a) - \ldots, \text{ etc.} \tag{34}$$

Applying this result to (32), in that equation we replace

$$\Sigma_4(w_A) \text{ by } \Sigma_{\sqrt{2}a,N}(w), \quad \text{with } N = 4,\ \beta = \pi/N,$$
$$\left.\begin{array}{l}\Sigma_4(w_I) \text{ by } \Sigma_{2a,N}(w),\\ \Sigma_4(w_1) \text{ by } \Sigma_{a,N}(w),\end{array}\right\} \quad \text{with } N = 4,\ \beta = 0.$$

Then its right-hand side is seen to be equivalent to

$$20w_0 + 4[A_0(2a) + A_4(2a) + A_8(2a) + \ldots$$
$$+ 2\{A_0(\surd 2a) - A_4(\surd 2a) + A_8(\surd 2a) - \ldots\}$$
$$- 8\{A_0(a) + A_4(a) + A_8(a) + \ldots\}],$$

where as in Part III, if terms of order a^6 are neglected,

$$A_0(a) = w_0 + \frac{a^2}{4}(\nabla^2 w)_0 + \frac{a^4}{64}(\nabla^4 w)_0,$$
$$A_4(a) = ka^4, \quad k \text{ being an unknown constant,}$$
$$A_8(a) = A_{12}(a) = \ldots \text{etc.} = 0,$$

and corresponding expressions hold for $A_0(2a)$, $A_4(2a)$, ..., etc. Consequently to this approximation the right-hand side of (32)

$$= 4w_0(5+1+2-8) + a^2(\nabla^2 w)_0\{2^2 + 2(\surd 2)^2 - 8\}$$
$$+ \frac{a^4}{16}(\nabla^4 w)_0\{2^4 + 2(\surd 2)^4 - 8\} + 4ka^4\{2^4 - 2(\surd 2)^4 - 8\}$$
$$= a^4(\nabla^4 w)_0, \quad \text{so that (32) is confirmed.}$$

An exactly similar treatment serves to establish (33).

16. Finite-difference approximations to $\partial^2/\partial x^2$ and $\partial^2/\partial y^2$ can be deduced in the usual way from Taylor's series. We have in figure 3, x, y being the co-ordinates of O,

$$w_1 + w_3 - 2w_0 = (w)_{x+a} + (w)_{x-a} - 2w_x = 2\left[\frac{a^2}{2!}\frac{\partial^2 w}{\partial x^2} + \frac{a^4}{4!}\frac{\partial^4 w}{\partial x^4} + \ldots\right]_x,$$

therefore with neglect of a^4 and higher powers of a

$$\left.\begin{aligned} &a^2\frac{\partial^2 w}{dx^2} \text{ may be replaced by } w_1 + w_3 - 2w_0,\\ \text{and (similarly)}\quad &a^2\frac{\partial^2 w}{\partial y^2} \text{ may be replaced by } w_2 + w_4 - 2w_0.\end{aligned}\right\} \tag{35}$$

Again, from Taylor's series we have

$$w_A - w_B = [w_{x+a} - w_{x-a}]_{y+a} = 2\left[a\frac{\partial w}{\partial x} + \frac{a^3}{3!}\frac{\partial^3 w}{\partial x^3} + \ldots\right]_{x,y+a}, \tag{i}$$

therefore with neglect of a^3 and higher powers of a

$$\left.\begin{aligned} &2a\left(\frac{\partial w}{\partial x}\right)_2 \text{ may be replaced by } w_A - w_B,\\ \text{and (similarly)}\quad &2a\left(\frac{\partial w}{\partial x}\right)_4 \text{ may be replaced by } w_D - w_C \text{ (figure 3).}\end{aligned}\right\} \tag{ii}$$

Also we have as in (i)

$$\left(\frac{\partial w}{\partial x}\right)_2 - \left(\frac{\partial w}{\partial x}\right)_4 = 2\left[a\frac{\partial}{\partial y}\left(\frac{\partial w}{\partial x}\right) + \frac{a^3}{3\,!}\frac{\partial^3}{\partial y^3}\left(\frac{\partial w}{\partial x}\right) + \ldots\right]_0, \tag{iii}$$

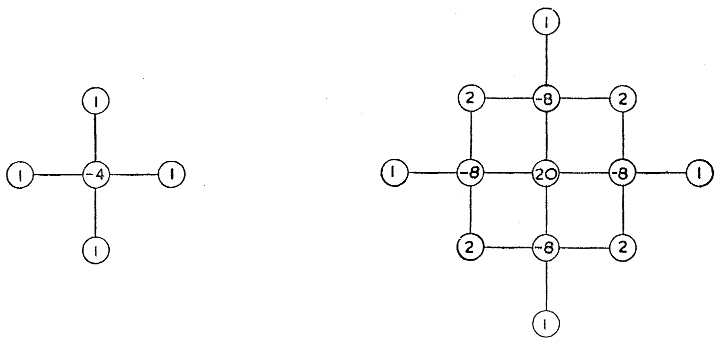
therefore according to (ii), with neglect of a^4 and higher powers of a

$$4a^2\left(\frac{\partial^2 w}{\partial x\,\partial y}\right)_0 \text{ may be replaced by } w_A - w_B - w_D + w_C. \tag{36}$$

Corresponding approximations can be deduced for triangular nets ($N = 6$). But equations which contain the operators $\partial^2/\partial x^2$, $\partial^2/\partial y^2$, $\partial^2/\partial x\,\partial y$ are best treated with a use of square-mesh nets.

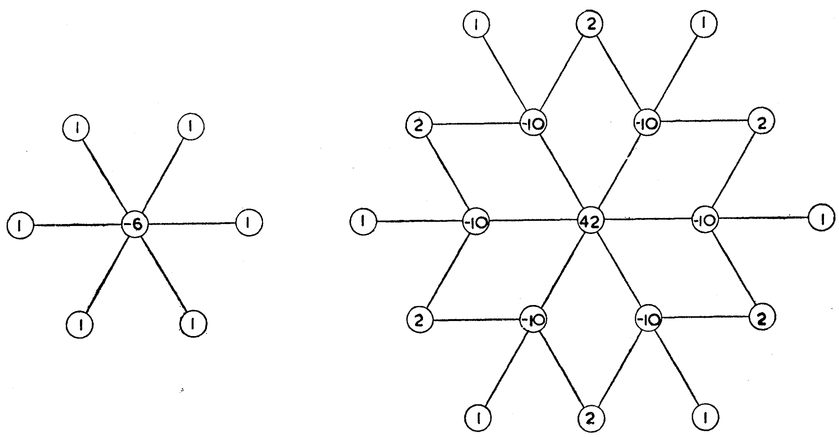
'Relaxation patterns'

17. Having these finite-difference approximations we can, in the manner of Part III, §§ 10–11, deduce from them the consequences of a unit displacement imposed on any nodal point. The results may be embodied in 'standard relaxation patterns'



(*a*) (*b*)

Relaxation patterns for the operators ∇^2 and ∇^4: square net ($N = 4$)



(*c*) (*d*)

Relaxation patterns for the operators ∇^2 and ∇^4: triangular net ($N = 6$)

Figure 5

whereby 'residual forces' can be liquidated, 'group relaxations' being derived from these as each problem may require.

The 'patterns' corresponding with (35) and (36) will be applied to 'Mathematical Problem I' *in combination,* therefore are not worth recording separately. Figure 5 compares, for nets of square and triangular mesh ($N = 4$ and 6), the relaxation patterns corresponding with the plane-harmonic operator ∇^2 and with the biharmonic operator ∇^4.

The use of interlacing nets

18. As in the treatment of Laplace's or Poisson's equation (Part III, § 13), labour can be saved by performing an initial liquidation of the transverse loading on a net of coarse mesh, then using the results to obtain a trial solution which is made the starting point of another liquidation with the use of a finer net. Errors in a trial solution will be corrected in the subsequent computation, therefore are not important except as increasing the subsequent labour: consequently no particular procedure is obligatory in this 'advance to a finer net',—practical convenience is the sole criterion.

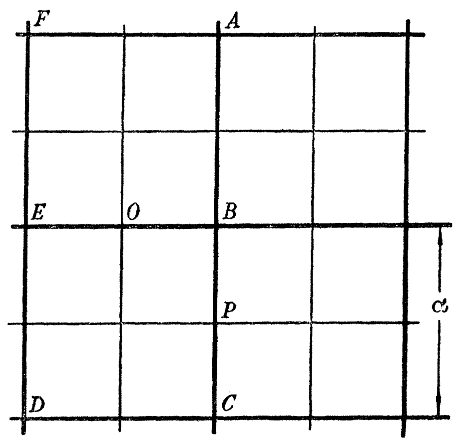


FIGURE 6

Relaxation being a more complicated process in biharmonic than in plane-harmonic problems, as much use as possible should be made of coarse nets in the early stages of computation. In some instances we have found it useful to begin with two independent calculations on nets of the largest mesh, arranging the two nets to interlace as shown in figure 6. Such treatment goes some way to neutralize the gravest disadvantage of a coarse net, namely, that it cannot take account of fine detail in the specified distribution of *edge* traction or displacement. Comparing the two solutions we can give weight to one or other as may be thought desirable, and the accepted values can be combined to give starting values for a finer net, nearer to the correct values than the results for either coarse net taken by itself.

The device can equally be applied to problems of the kind discussed in Part III, and conversely, the procedure of that paper is applicable (and has been applied) to some of the problems of this paper.

Treatment of the double boundary condition

19. In 'Mathematical Problem I' (§ 11), the governing equations have the form of (10) expressed in purely numerical form (cf. § 12), and the boundary conditions fix the values of u and v. On account of the terms in Δ both u and v appear in both of equations (10), otherwise these would present two independent cases of Poisson's

equation. Orthodox treatment is difficult, and little progress has been made: for a 'relaxation' treatment no extension of technique is needed except a keeping account of u and v *simultaneously* on two similar nets, and this has been found quite easy.

In 'Mathematical Problem II', on the other hand, orthodox analysis finds less difficulty but the double boundary condition presents new obstacles to a relaxation attack. We have to solve (3) in respect of some specified function W, also to satisfy boundary conditions which fix the values of $\partial w/\partial x$ and $\partial w/\partial y$, W, w, x, y being purely numerical. This is a new problem, and it calls for a new technique to replace the approximate treatment of 'irregular stars' (Part III, §§ 23–4).

20. In case B of the extensional problem (§ 7) the boundary values of $\partial\chi/\partial x$, $\partial\chi/\partial y$ and therefore χ can be expressed in terms of the specified edge tractions, and in case A of the flexural problem boundary values of w, $\partial w/\partial x$ and $\partial w/\partial y$ are known since w and $\partial w/\partial \nu$ are specified. Working with a net of finite mesh, we must use these data to impose values on χ or w at nodal points just inside and just outside the boundary.

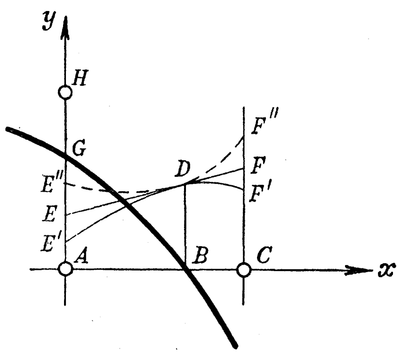


FIGURE 7 *a*

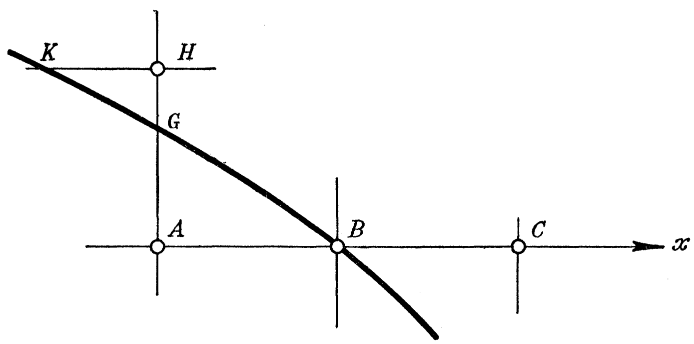


FIGURE 7 *b*

In figure 7 *a* these nodal points are indicated by open circles, and the boundary by a bold line; at B, on the boundary, we know the values of w and of $\partial w/\partial x$. Representing w_B by BD, $(\partial w/\partial x)_B$ by the slope of EDF, we see that AE, CF will give an approximate indication of w_A, w_C, but that higher differentials of w will result in higher or lower values as suggested by AE', CF' or by AE'', CF''. At the start we do not know these higher differentials, consequently have no alternative to working with the values given by AE, CF; but some second differentials, at least, will attain their greatest values at the boundary, so it must be expected (at least, in the early stages) that our starting values of w will be sensibly inaccurate, and it is useless to carry liquidation more than a short way until they have been amended.

Figure 7 *b* exhibits a special case in which B falls exactly on the boundary. Similar considerations apply: we can estimate values of w both for A and for C, but with little confidence. At points whose distances from the boundary—measured along a 'string' of the chosen net—are not greater than the mesh size a, values of w will be dictated by the boundary conditions; but unless a is very small they will be given neither precisely nor uniquely (as is evident when it is realized that w_A can equally well be estimated from the values of w and of $\partial w/\partial y$ at G).

21. In these circumstances the following procedure is indicated: Start with values estimated as above from the boundary values of w, $\partial w/\partial x$, $\partial w/\partial y$ *without allowance for higher differentials*; carry liquidation a little way on the basis of these starting values; then correct them in the light of the evidence so obtained. For example, if values w_K, w_L, w_M are obtained by a partial liquidation, a rough plotting (figure 8) shows that the curve of w which passes through E and F (viz. $KLMEF$) will not in fact touch EF at D: consequently the correct curve may be expected to have a form of the kind shown in dotted lines through $KLmeDf$, and the modified values given by $M'm$, Ae, Cf may be taken as a basis for further liquidation *in which points A and C are not relaxed*. By this means greater advantage can be taken of coarse nets in the early stages.

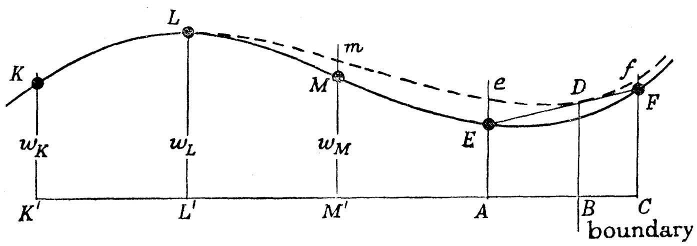


FIGURE 8

22. Two points of detail should be made: First, in the determination of w_A and w_C the second may be expected to dominate the third and higher differentials, consequently in figure 8 the ratio of Ee/Ff should approximate to $(AB/BC)^2$. Secondly, whereas for A (figure 7 *b*) the boundary conditions yield two values of w which call for a compromise (§ 20), the two values similarly given for the value at an external node H *can both be accepted*, since there is no necessity for w (or χ) to be single-valued at points outside the boundary.

Points like A, C and H (figures 7) correspond with boundary points in the problems of Part III, in that their 'displacements' are restricted by the boundary conditions and in consequence the residual forces on them need not be liquidated. The region in which they lie (an irregular strip which includes the boundary) will henceforth be termed the 'selvedge' of the net in question, and points like A or like C and H will be termed internal or external 'selvedge points'.

PRELIMINARY ELIMINATION OF SINGULARITIES

23. As in Part III (§ 5) we contemplate that singularities in the specified loading or boundary tractions can be eliminated *initially* by recourse to the principle of superposition. In two dimensions, analytical solutions exist for the effects of a concentrated force acting either at or inside the boundary (cf. Love 1927, §§ 147–52): using these we

can transform our problem into one which does not involve singularities, therefore is more suited to the relaxation approach. The device was not in fact employed in any of the problems of Part III, but in Part V (e.g. § 5) it was used to dispose of a logarithmic infinity which was encountered in a problem of conformal transformation. Here its application will entail more labour, since the analytical solutions are more complex; but its principles are unchanged whether the singularity is on or inside the boundary, so do not call for further description here.

III. PRACTICAL DETAILS OF THE RELAXATION TECHNIQUE

24. We now describe in detail, and in relation to particular examples, the relaxation procedure by which approximate solutions can be obtained. Four separate problems will be considered—viz. one example of each of the cases listed in § 8. First we shall exemplify case A of the extensional problem.

'Mathematical Problem I.' An extensional example

Giving to A in (10) the value $1-2\sigma$, we have equations governing the component displacements u and v under conditions of plane strain. Here we shall assume for σ the value 0·3 (a representative figure for steel), so that $A = 0{\cdot}4$. (Under conditions of plane stress, the description which follows will still apply, but (cf. § 6) the somewhat different value $(1-\sigma)/(1+\sigma) = 7/13$ must be given to A.)

Eliminating dimensional factors in the manner of § 12, we replace (10) by (10A) or (on suppression of the dashes, and with $A = 0{\cdot}4$ as above) by

$$\left.\begin{aligned}\left[1{\cdot}4\frac{\partial^2}{\partial x^2}+0{\cdot}4\frac{\partial^2}{\partial y^2}\right]u+\frac{\partial^2 v}{\partial x\,\partial y}+0{\cdot}4X=0,\\ \frac{\partial^2 u}{\partial x\,\partial y}+\left[0{\cdot}4\frac{\partial^2}{\partial x^2}+1{\cdot}4\frac{\partial^2}{\partial y^2}\right]v+0{\cdot}4Y=0,\end{aligned}\right\}\tag{37}$$

x, y, u, v, X, Y being now purely numerical. Next, with a view to Relaxation Methods we replace the differentials in (37) by their finite-difference approximations as given in (35) and (36). Then, in the terminology of those methods, to satisfy (37) approximately on a rectangular net of mesh-size La we must liquidate 'residual forces' acting at the nodal points and given by

$$\mathbf{F}_x = \mathsf{F}_x + F_x, \quad \mathbf{F}_y = \mathsf{F}_y + F_y,$$

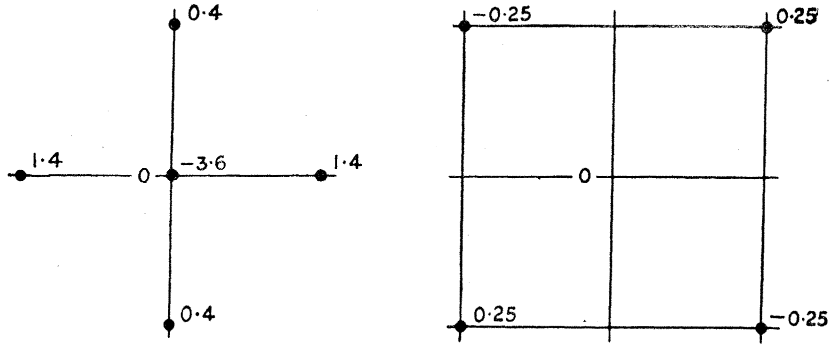
where, for any point 0, figure 3,

$$\mathsf{F}_x = 0{\cdot}4a^2X, \quad \mathsf{F}_y = 0{\cdot}4a^2Y$$

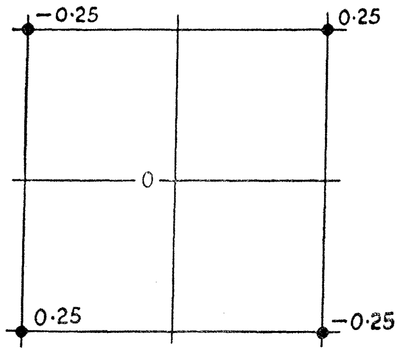
are the 'initial forces' and

$$\left.\begin{aligned}F_x &= 1{\cdot}4(u_1+u_3)+0{\cdot}4(u_2+u_4)-3{\cdot}6u_0+0{\cdot}25(v_A-v_B+v_C-v_D)\\ F_y &= 0{\cdot}25(u_A-u_B+u_C-u_D)+0{\cdot}4(v_1+v_3)+1{\cdot}4(v_2+v_4)-3{\cdot}6v_0\end{aligned}\right\}\tag{38}$$

are the forces consequent on displacements u, v occurring at 0 and at surrounding points. For the computations we shall require 'relaxation patterns' (§ 17) giving the changes produced in $\mathbf{F}_x$, $\mathbf{F}_y$ by an isolated displacement u or v imposed at any one nodal point. These are easily deduced from the last two of (38), and have the forms shown in figure 9. There are two 'patterns' relating to u and two to v.



(a) Effects on $\mathbf{F}_x$, $\mathbf{F}_y$ of an isolated displacement $u = 1$ at 0



(b) Effects on $\mathbf{F}_x$, $\mathbf{F}_y$ of an isolated displacement $v = 1$ at 0

FIGURE 9

25. Using the patterns, and keeping account of u and v simultaneously on the same or on two similar nets, it is easy to liquidate specified forces $\mathbf{F}_x$, $\mathbf{F}_y$ in the manner of Part III. Since the boundary values of u and v are specified, points on the boundary must not be displaced in the liquidation process.

We now describe our solution of the problem shown in figure 10, where (in non-dimensional notation, and when L in (29) denotes the length of the shorter side) the boundary conditions are

$$\left.\begin{aligned} u = v &= 0 \text{ at all points on the sides } x = \pm\tfrac{3}{2}, \\ u = v &= 0 \text{ at all points on the sides } y = -\tfrac{1}{2}, \\ u = 0,\ v \times 10^4 &= 4x^2 - 9, \text{ on the side } y = +\tfrac{1}{2}, \end{aligned}\right\} \tag{39}$$

and the body forces are given by

$$X = 0, \quad Y = -10^{-6}, \tag{40}$$

—this last figure being representative of gravity acting on a steel plate for which $L = 100$ cm. By treating separately the effects of the body forces and of the imposed displacements we obtain solutions which can be superposed in any proportion. Multiplying factors were introduced with the aim of avoiding decimal points. On account of symmetry, only one half of the plate had to be considered.

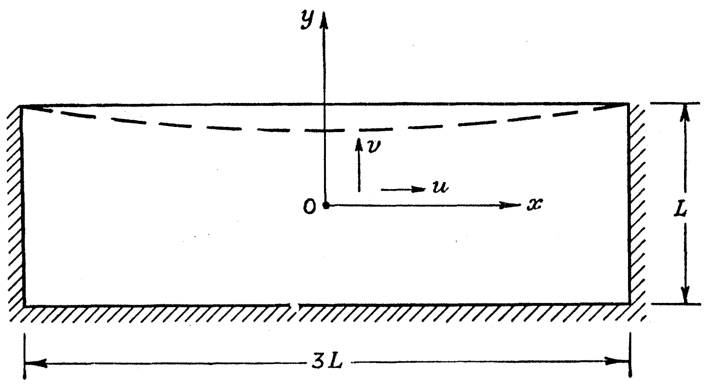


FIGURE 10

EFFECT OF IMPOSED EDGE DISPLACEMENTS

26. Here (figure 11) we multiplied the imposed displacements by 10^7 to obtain edge values -9000, -8000, -5000 for v, and we started with a net for which $a = \frac{1}{2}$. Only five points (in the whole plate) can be moved, so the first approximation was soon obtained. The values shown were computed in about 10 minutes, and are such that a change of 1 in the last significant figure would entail increased magnitudes for the residual forces.

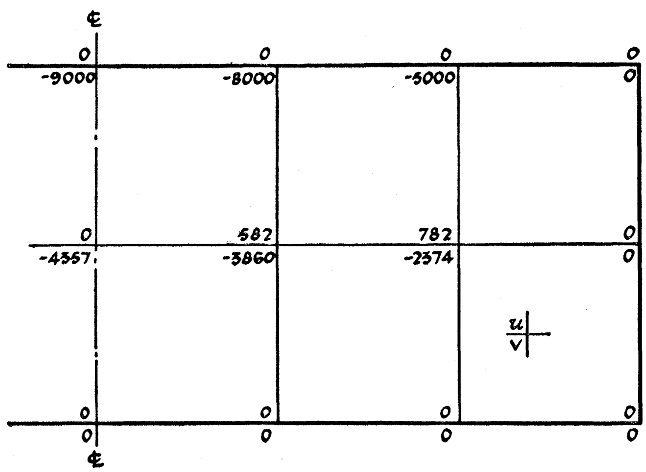


FIGURE 11

Next (figure 12) we reduced the mesh-size a to $\frac{1}{4}$ and inserted additional edge values, then by rules which will be explained in § 28 we derived from their values in figure 11 starting approximations to the values of u, v at intermediate nodal points, and liquidated these with the aid of the standard 'patterns' of figure 9. Less than 1 hour's work resulted in the values shown, which satisfy the same criterion of accuracy as before. The v-values have been only slightly altered in this 'advance to a finer net'.

In the third and final net, a was further reduced to $\frac{1}{8}$, and a liquidation process occupying less than 10 hours gave the values which are recorded in figure 13. No new

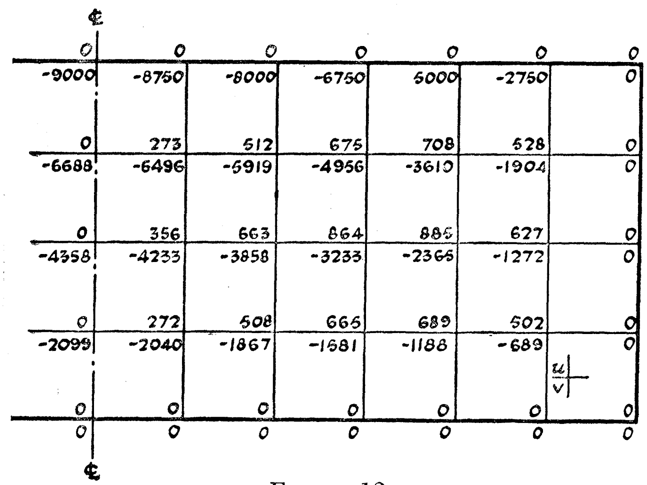


FIGURE 12

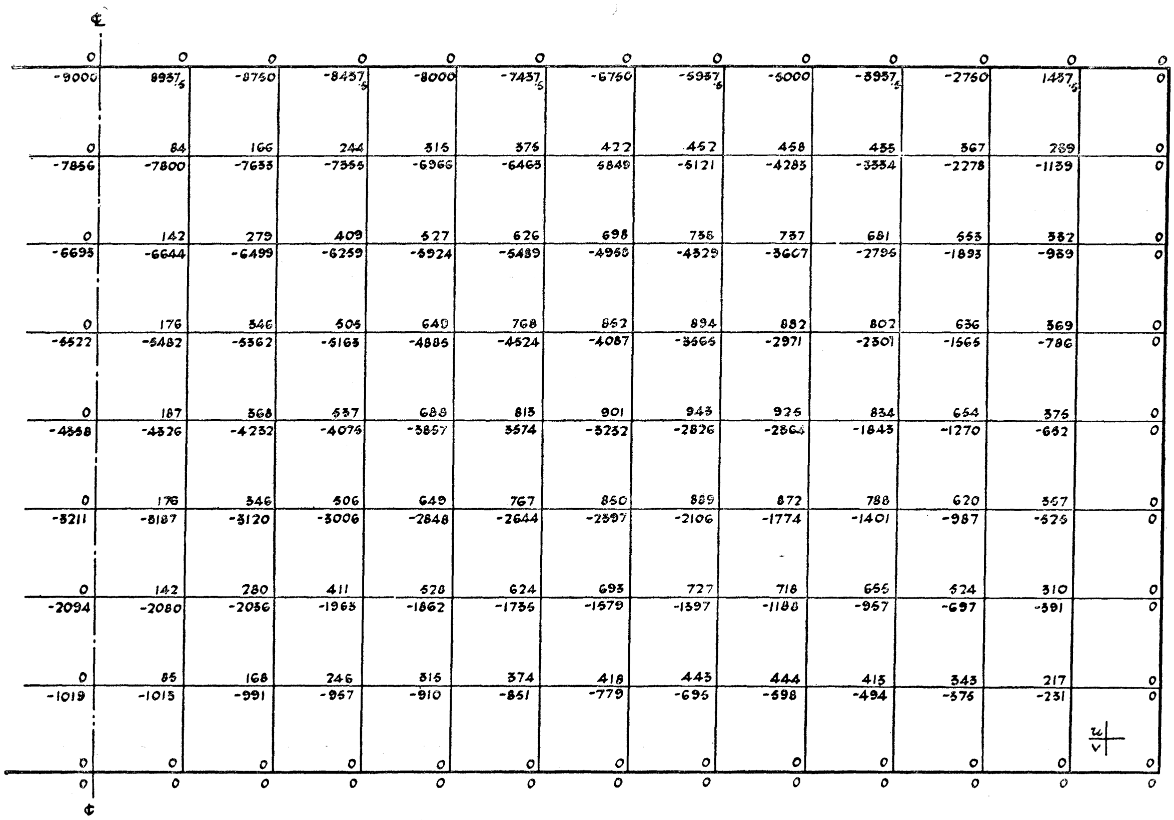


FIGURE 13. Accepted solution of the extensional problem of Fig. 10: imposed edge displacements.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

feature calls for mention here. In all three diagrams the recorded numbers are values of u and v *multiplied by* 10^7.

Effect of imposed body forces

27. The work in this second part of the problem was exactly similar, and it will suffice to give the final results for a net in which $a = \frac{1}{8}$ (figure 14). A multiplying factor 10^9 was imposed in order to obviate decimals, so the recorded numbers are values of u and v multipled by 10^9. As was expected, the distortion produced by gravity (in a plate of which the shorter side $L = 100$ cm. Cf. § 25) proved to be very small in relation to the effect of imposing the edge displacements (39).

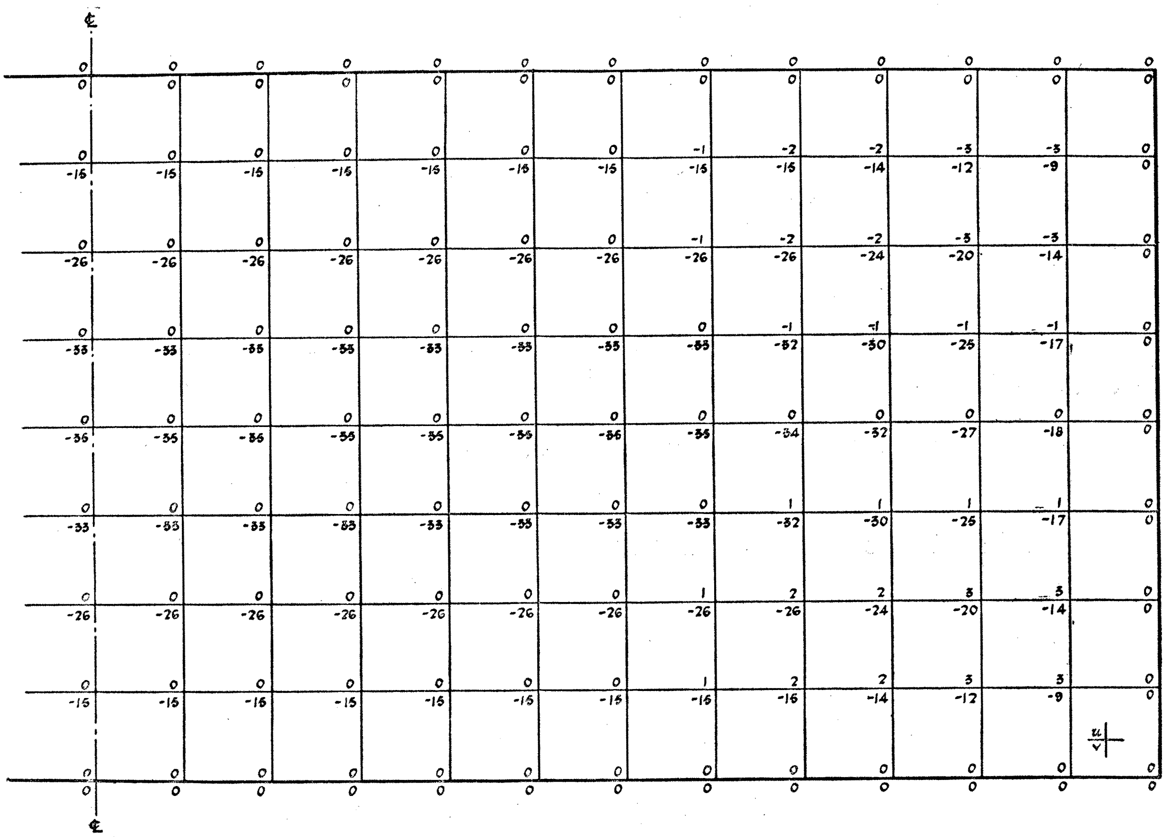


Figure 14. Accepted solution of the extensional problem of Fig. 10: imposed body forces.

The 'advance to a finer net'

28. At every stage in these computations except the first, we start having values already estimated for nodal points of a coarser net, but needing values at those nodal points which now for the first time come into the picture. Any choice of values will leave some residual forces requiring liquidation, therefore in a sense it does not matter how the choice is made; but a well-judged choice is desirable as entailing less labour in subsequent liquidation. In Part III it was considered in relation to the single operator

∇^2: here the governing equations involve the three distinct operators $\partial^2/\partial x^2$, $\partial^2/\partial y^2$, $\partial^2/\partial x\partial y$.

Given values of u and v at nodal points of the coarse net indicated by bold lines in figure 6, we can deduce their values at the centre of every mesh by making $\mathbf{F}_x$, $\mathbf{F}_y$ zero according to (38) with a replaced by $\frac{1}{2}a$, provided that we have their values for the other nodal points of the finer net (of mesh-size $\frac{1}{2}a$), which are the middle points of mesh-sides of the coarse net. Here a different procedure is required, and in treating this example we proceeded as follows:

At O in figure 6 (for example) we used the approximations

$$\left.\begin{aligned} \frac{a^2}{4}\left(\frac{\partial^2 u}{\partial x^2}\right)_O &\approx u_E+u_B-2u_O,\\ 2a^2\left(\frac{\partial^2 v}{\partial x\partial y}\right)_O &\approx v_A-v_F-v_C+v_D,\\ 2a^2\left(\frac{\partial^2 u}{\partial y^2}\right)_O &\approx a^2\left\{\left(\frac{\partial^2 u}{\partial y^2}\right)_E+\left(\frac{\partial^2 u}{\partial y^2}\right)_B\right\} \approx (u_F+u_D-2u_E)+(u_A+u_C-2u_B), \end{aligned}\right\} \tag{41}$$

with similar approximations for $(\partial^2 v/\partial x^2)$, etc. The first of (41) conforms with (35), a being replaced by $\frac{1}{2}a$; the second is obtained like (36), a being replaced by $\frac{1}{2}a$ in (ii) *but not in* (iii) of § 16; and the third is a deduction from the second of (35). Substituting in (37), we obtain

$$\left.\begin{aligned} 5\cdot6(u_E+u_B-2u_O)+0\cdot2(u_A+u_C-2u_B+u_D+u_F-2u_E)&\\ +\tfrac{1}{2}(v_A-v_F-v_C+v_D)+0\cdot4a^2X&=0,\\ \tfrac{1}{2}(u_A-u_F-u_C+u_D)+0\cdot7(v_A+v_C-2v_B+v_D+v_F-2v_E)&\\ +1\cdot6(v_E+v_B-2v_O)+0\cdot4a^2Y&=0, \end{aligned}\right\} \tag{42}$$

a denoting the mesh-size of the coarser net. From these relations u_O, v_O (figure 6) can be calculated; the other points of types O and P can be treated similarly; and then, having trial values of u and v, we can use (38) to calculate the initial force at every nodal point of the finer net.

Though somewhat elaborate, this procedure (since it employs the best possible approximations) reduces to a minimum the labour of subsequent liquidation. In this instance it was regarded as justified by results.

'Mathematical Problem I.' A flexural example

29. Cases of flexure under imposed edge tractions are not of frequent occurrence in practice, consequently it is not easy to devise an interesting flexural example of 'Mathematical Problem I'. Figure 15 suggests a form of test-piece, simple to manufacture, which might be used to determine the elastic properties in bending of a sample of flat

plate. Couples applied at the ends will be magnified by the constriction to give stresses at the 'waist' which are relatively high and (by Saint Venant's principle) practically independent of the manner in which the edge-couples are distributed. Consequently the stresses can be computed (within the elastic limit) on any reasonable assumption regarding that distribution, and the results will serve to interpret the experimental measurements.

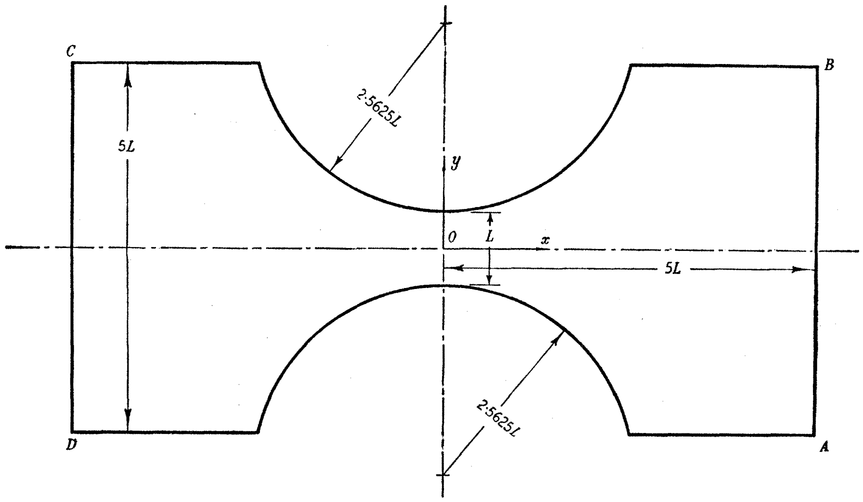


FIGURE 15

Here we assume uniform line-intensity of flexural couple (G) along the edges AB, CD in figure 15, H and N having zero values along those edges, and all of G, H, N vanishing elsewhere on the boundary. Then, in §§ 9–10, since no transverse loading (Z) is operative the function $\nabla^2 w_1$ must vanish everywhere, so that X_1, Y_1, in (24), are zero according to (25). Also we have from (19)

$$\left.\begin{aligned} &A = 0, \text{ everywhere,} \\ &DB = -G \text{ along } AB, \ +G \text{ along } CD, \ 0 \text{ elsewhere on the boundary,} \end{aligned}\right\} \qquad \text{(i)}$$

and hence from (20), *G being uniform by assumption*,

$$\left.\begin{aligned} U &= 0, \text{ everywhere,} \\ V &= -\frac{G}{D} y \text{ along } AB, \ CD, \\ &\mp \frac{1}{2}\frac{G}{D}\overline{AB} \text{ at all points in } BC, \ DA. \end{aligned}\right\} \qquad \text{(ii)}$$

30. To eliminate dimensional factors we write

$$x = L.x', \quad y = L.y', \quad U = \frac{LG}{D} U', \quad V = \frac{LG}{D} V', \qquad (43)$$

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

L now denoting the width of the specimen at its waist. Then the boundary conditions (ii) require that

$$\left.\begin{aligned} U' &= 0, \text{ everywhere,} \\ V' &= -y' \text{ along } AB, CD \\ &= -2{\cdot}5 \text{ at all points in } BC \\ &= +2{\cdot}5 \text{ at all points in } DA \text{ (figure 15),} \end{aligned}\right\} \tag{44}$$

since $AB = 5L$; and from (24), in which $X_1 = Y_1 = 0$, we deduce that

$$\frac{\partial \Delta'}{\partial x'} + A\nabla^2 U' = 0, \quad \frac{\partial \Delta'}{\partial y'} + A\nabla^2 V' = 0, \tag{45}$$

Δ' now standing for $\dfrac{\partial U'}{\partial x'} + \dfrac{\partial V'}{\partial y'}$, and A having the value $\dfrac{1+\sigma}{1-\sigma}$ which is 2 when $\sigma = \frac{1}{3}$, *as we shall here assume.*

From (27), suppressing $\nabla^2 w_1$, we have

$$-G\frac{\partial V'}{\partial y'}, \quad -G\frac{\partial U'}{\partial x'}, \quad -\tfrac{1}{2}G\left(\frac{\partial U'}{\partial y'} + \frac{\partial V'}{\partial x'}\right) \tag{46}$$

as expressions for the stress-couples G_x, G_y, H_{xy} at any point in the plate; and the transverse deflexion w can be calculated if required, since (cf. § 10) the three quantities (46), divided by D, are the values of

$$-\left(\frac{\partial^2 w}{\partial x^2} + \sigma\frac{\partial^2 w}{\partial y^2}\right), \quad -\left(\frac{\partial^2 w}{\partial y^2} + \sigma\frac{\partial^2 w}{\partial x^2}\right), \quad (1-\sigma)\frac{\partial^2 w}{\partial x\,\partial y}.$$

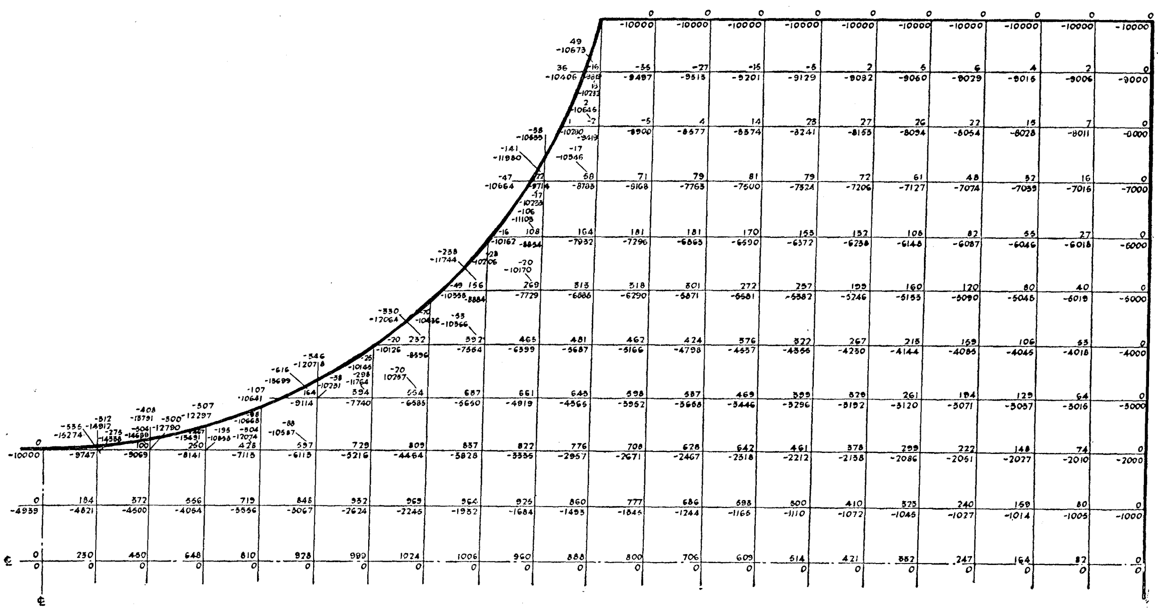


FIGURE 16. Accepted values of U and V ($\times 4000$) for the flexural problem of figure 15.

31. Comparing (45) with (10), we have confirmation of the analogy demonstrated in §§ 9–11. Interpreted 'extensionally', our problem is to calculate the displacements U, V which result when specified displacements are imposed on the boundary of the test-specimen (figure 15), and in the example now under consideration the imposed distortion (44) has the nature of a lateral compression. There are no body forces.

Our final results for this problem (on a net of mesh-size $a' = a/L = \frac{1}{4}$) are presented in figure 16. The same technique was employed as in §§ 24–28, except that here, at points near the curved part of the boundary, we had to deal with 'irregular stars'. These were discussed in §§ 23–4 of Part III, but only in relation to the single operator ∇^2: here they call for discussion in relation to all three of the approximations (41).

The treatment of 'irregular stars'

32. Part III proceeded on the assumption (a logical consequence of replacing continuous 'membranes' by nets of finite mesh) that every string remains straight when a net is deflected, and that the transverse force which it exerts is directly proportional to its slope. Now this slope, when its ends are displaced through given distances, will be inversely proportional to the length of the string: consequently, in the formulae for residual forces, the force exerted by a string of standard mesh-length a was increased in the ratio $1/x$ when the string had length xa.

The same assumption may be utilized somewhat differently as follows: In figure 17, the nodal points numbered 3 and 0 are supposed to lie inside the boundary, but the string 01′ meets the boundary at 1′, *where accordingly the deflexion is specified*. Let the distance 01′ be denoted by ξa, and let the string be continued (as shown in broken lines) to a point 1, outside the boundary, whose distance from 0 is a. Then (f standing indifferently for u or v) the relation

$$\xi(f_1 - f_0) = f_1' - f_0 \tag{47}$$

will give a value f_1, to be attached to f at the boundary point 1′, which can be inserted in the first of (35). Similar treatment can be applied to the second of (35), but (36) presents a new problem.

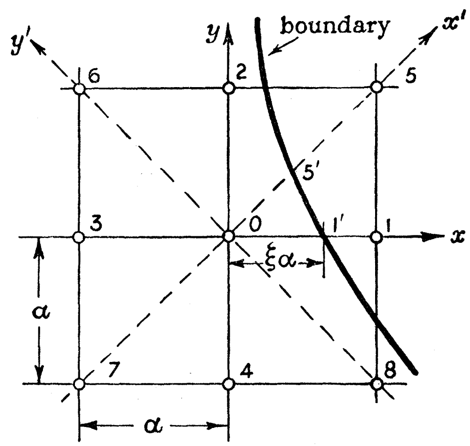


FIGURE 17

Transforming the axes of co-ordinates by counter-clockwise rotation through 45° (figure 17), it is easy to show that

$$4a^2 \frac{\partial^2 f}{\partial x \partial y} \equiv 2a^2 \left(\frac{\partial^2 f}{\partial x'^2} - \frac{\partial^2 f}{\partial y'^2}\right), \tag{i}$$

and consistently with the first and second of (35) we may replace

$$\left.\begin{array}{lll} 2a^2\dfrac{\partial^2 f}{\partial x'^2} & \text{by} & f_5+f_7-2f_0, \\ 2a^2\dfrac{\partial^2 f}{\partial y'^2} & \text{by} & f_6+f_8-2f_0, \end{array}\right\} \quad \text{(ii)}$$

since the distance of 0 from 5, 6, 7 and 8 is $a\sqrt{2}$. Hence

$$4a^2\frac{\partial^2 f}{\partial x\,\partial y}=f_5-f_6+f_7-f_8, \quad \text{(iii)}$$

confirming (36); and the same device as before can be employed to deal with a case in which (e.g.) the 'string' from 0 to 5 is cut by the boundary at a point 5′, except that ξ must now express the length as a fraction of $a\sqrt{2}$.

Sketching may be employed in the manner of §§ 20–1 to improve on the approximation of (47). As remarked in § 22, double or multiple values of U or V can be accepted *at points outside the boundary*: figure 16 shows as many as three pairs of values (for U, V) attached to certain points.

'Mathematical Problem II.' A flexural example

33. Here too as a first illustration we take a fairly simple example, this time of case A of the flexural problem. We shall investigate the transverse deflexion under uniform pressure of a clamped square plate—a system having 8-fold symmetry (viz. symmetry with respect to both medians and both diagonals). There are no singularities calling for preliminary elimination in the manner of § 23.

The governing equation is (5) of § 4, Z being now specified as independent of x and y: thus Z_0 in (28) is identical with Z, and on elimination of dimensional factors in the manner of § 12 (L now denoting the length of a side of the plate) we reduce the governing equation to

$$\nabla^4 w = 1, \quad (48)$$

and the boundary conditions to

$$\frac{\partial w}{\partial x}=0, \quad \frac{\partial w}{\partial y}=0, \quad \text{on the edges } x=\pm\tfrac{1}{2}, \quad y=\pm\tfrac{1}{2}, \quad (49)$$

when the origin is situated at the centre of the plate. x, y and w are now purely numerical (they are the quantities which in (28) and (3 A) were denoted by x', y', w').

34. On account of symmetry only one quarter of the plate need be considered, and further simplification comes from the fact that the boundaries are straight lines containing rows of nodal points. By displacing simultaneously, and through equal distances, a nodal point adjacent to the straight boundary *and its image point with respect to that boundary* we automatically satisfy the condition $\partial w/\partial \nu = 0$.

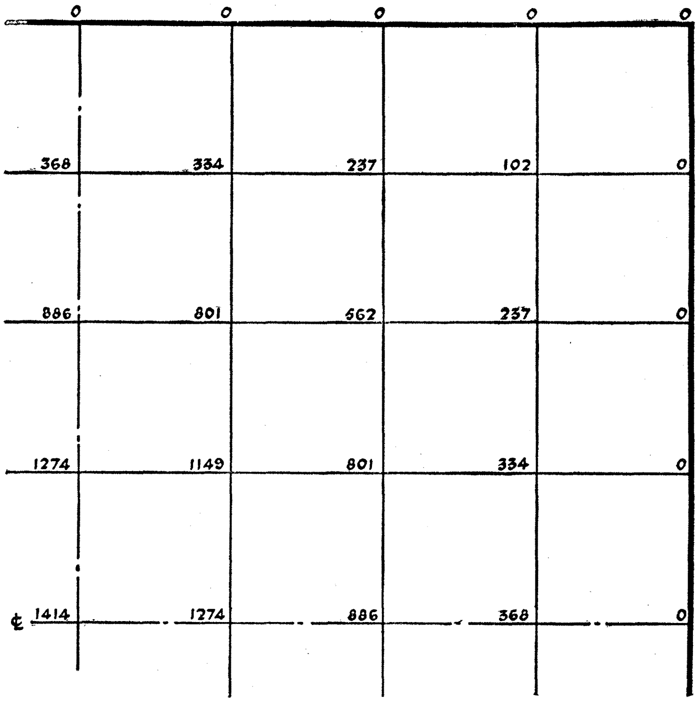


FIGURE 18

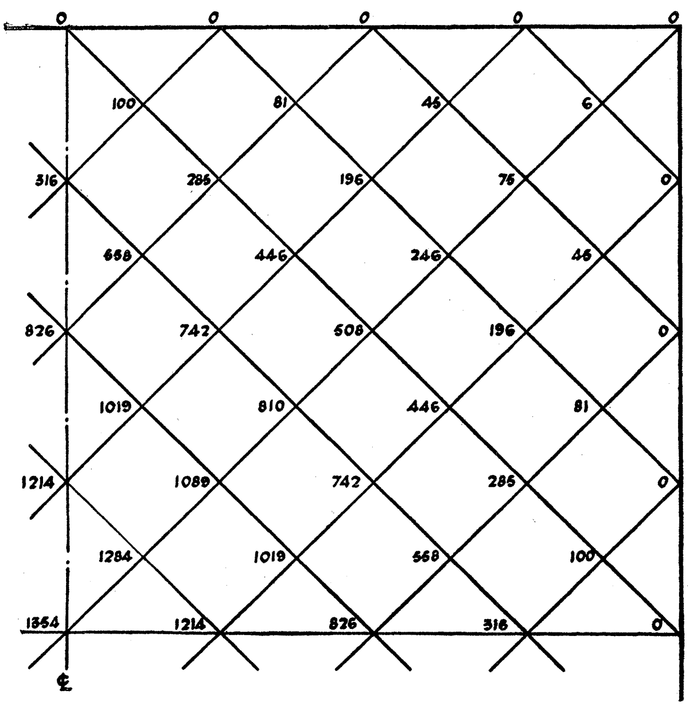


FIGURE 19

To eliminate decimals all forces and displacements were multiplied by 10^6. Liquidation was started on a net of side $a = \frac{1}{8}$ (figure 18) with the use of the standard 'relaxation pattern' of figure 5 (*b*). On its completion, $\nabla^2 w$ was calculated in accordance with (30), then from its four corner values for each mesh a central value was deduced, to be used as a starting assumption in the calculation of w-values for a second and finer net ($a = 1/8\sqrt{2}$: figure 19). This net was treated similarly to obtain starting values of w for the final net ($a = \frac{1}{16}$) of figure 20.

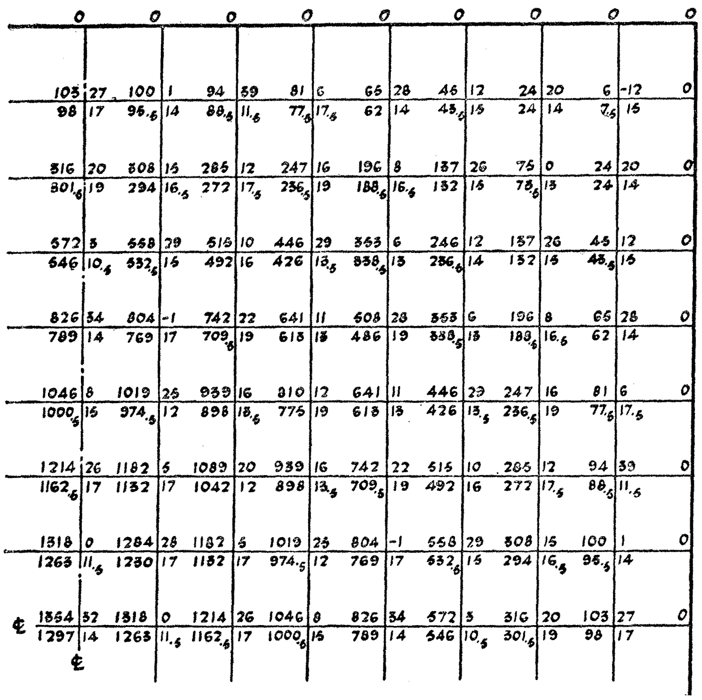


FIGURE 20. Starting and accepted values of w ($\times 10^6$) for the flexural problem of § 33.

At each advance to a finer net, the starting values of w (deduced from the previous liquidation in the manner described) were found to account for somewhat more than the whole of the specified loading, so were multiplied by an appropriate factor (e.g. 0·958 in the case of figure 19). Similar multiplication was introduced about half-way through the liquidation process summarized in figure 20. In that diagram both the starting and the final (i.e. accepted) values of w are recorded, together with the loadings for which they account. (The specified loading per nodal point, multiplied as above, is 15·26.) The accuracy is such that greater residual forces would result from a unit change in the last digit of any displacement.

'MATHEMATICAL PROBLEM II.' AN EXTENSIONAL EXAMPLE

35. Our last example concerns extensional stresses due to specified edge tensions, a problem of the class to which photo-elastic methods are most commonly applied (§ 1). It has in fact been so treated by Coker and Filon (1931, § 7·06).

Figure 21 records the dimensions of the B.S.I. standard 'briquette' for the testing of cement, with an indication (shaded) of the 'jaws' by which tensile loading is applied. The waisted form is intended to ensure fracture across the minimum section, and (on the ground that in this neighbourhood the boundary is not stressed) a definite stress-system is assumed to obtain at the waist, notwithstanding that the applied tractions are not known with great precision. Uncertainty is introduced (1) by friction accompanying the wedging action on the jaws, due to which the applied tractions may have

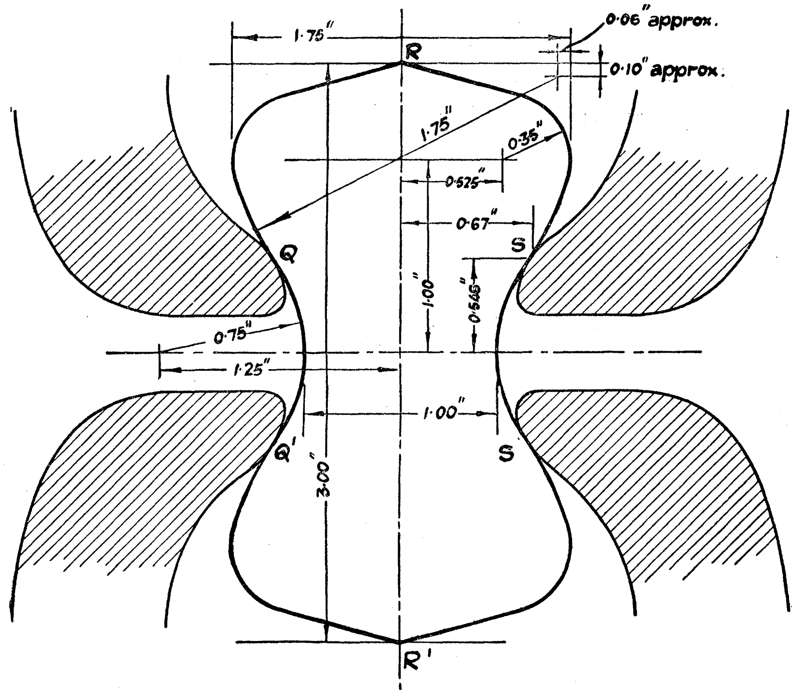


FIGURE 21. B.S.I. standard cement briquette.

horizontal components although their resultant is vertical, (2) by distortion of the briquette and of the jaws as a consequence of the pressure between them, owing to which the load distribution cannot be predicated exactly.

The photo-elastic technique employs a model briquette and actually performs a tensile test of this specimen between model 'jaws'; consequently it fails to resolve the first uncertainty, since it is hardly safe to assume that it reproduces the wedging action exactly. Here, using the greater freedom afforded by our theoretical approach, we shall examine the extent of that uncertainty by calculating two distinct stress-systems, viz. (a) the stresses induced when the applied tractions are everywhere vertical and (b) the *additional* stresses resulting from horizontal components which entail a pinching action on each enlarged end. (Within the range of Hooke's Law the two systems (a) and (b) may be combined in any proportion.)

We shall not attempt to resolve the second uncertainty, for the reason that the finer details of the load distribution must in any event be 'blurred' in computations which employ a net of finite mesh. Both in (*a*) and in (*b*) we shall assume the edge tractions to be concentrated at points; but our solutions will hold (approximately) in respect of edge tractions however distributed, provided that they have the correct resultant and are localized in a region between two 'strings' of the finest net.

36. The basic theory has been given in §§ 5–7. Ω_1 and Ω_2 vanish in the absence of body forces, and if Ox, Oy are respectively horizontal and vertical as in figure 21, then the specified edge tractions are:

In case (*a*) *of* § 35:

$X_\nu = 0$ everywhere,
$Y_\nu = 0$ except in the region of applied load;

In case (*b*) *of* § 35:

$X_\nu = 0$ except in the region of applied load,
$Y_\nu = 0$ everywhere.

On the assumption of concentrated forces, it will follow from (12) of § 7 that in case (*a*)*:

$\dfrac{\partial \chi}{\partial x} = 0$ in the end portions of the boundary,

$= \pm \frac{1}{2} T$ in the 'waist' between the points of loading, T denoting the total tension on the specimen, and the sign in the ambiguity being that of x,

$\dfrac{\partial \chi}{\partial y} = 0$, everywhere;

in case (*b*):

$\dfrac{\partial \chi}{\partial x} = 0$, everywhere,

$\dfrac{\partial \chi}{\partial y} = \mp P$ in the end portions of the boundary, P denoting the total pinching action on each end, and the sign in the ambiguity being opposite to that of y,

$= 0$ in the 'waist'.

The governing dimension of the briquette is the width of its narrowest section, i.e. its width on the line $y = 0$. Denoting this by L, we eliminate 'dimensional' factors by writing

$$\left.\begin{aligned} x &= Lx', & y &= Ly', \\ \text{with} \quad \chi &= TL\chi' \text{ in case } (a), & \chi &= PL\chi' \text{ in case } (b). \end{aligned}\right\} \tag{50}$$

Then, in case (*a*), such quantities as $\dfrac{\partial^2 \chi'}{\partial x'^2}$ measure stresses expressed as multiples of

* Any constant value can be attached either to $\partial\chi/\partial x$ or to $\partial\chi/\partial y$ without affecting the stresses (cf. § 7).

T/L, the average tensile stress at the minimum section; in case (b), they measure stresses expressed as multiples of P/L. The boundary conditions are:

in case (a):

$$\left.\begin{aligned}\frac{\partial\chi'}{\partial x'} &= 0 \text{ in the end portions, } = \pm\tfrac{1}{2} \text{ in the `waist'},\\ \frac{\partial\chi'}{\partial y'} &= 0, \text{ everywhere;}\end{aligned}\right\} \tag{51}$$

in case (b):

$$\left.\begin{aligned}\frac{\partial\chi'}{\partial x'} &= 0, \text{ everywhere,}\\ \frac{\partial\chi'}{\partial y'} &= \mp 1 \text{ in the end portions, } = 0 \text{ in the `waist'}.\end{aligned}\right\} \tag{52}$$

In (51) the sign in the ambiguity is that of x', in (52) it is that of $-y'$.

37. Boundary values of χ' will be required in subsequent work. A constant may be added so as to make χ' zero at any desired point, and in case (a) we shall make it zero in the end portions of the boundary (outside the points of loading): then $\chi' = \frac{1}{2}(x'_P - x'_C)$ at any point P in the 'waist' for which $x'_P > 0$, also at the image point of P with respect to the y'-axis. In case (b) we shall make χ' zero in the waist, therefore $-(y'_Q - y_C)$ at any point Q of the end portion for which $y'_Q > y_C$ and also at the image point of Q with respect to the x'-axis. Symmetry requires (in both cases) that $\partial\chi'/\partial y' = 0$ at all points of the x'-axis, $\partial\chi'/\partial x' = 0$ at all points of the y'-axis.

38. The singularities entailed by the concentrated forces can be eliminated in the manner of § 23 with the aid of the known solution (Michell 1900, p. 35; Love 1927, § 149)

$$\chi = \frac{r}{2\pi}(2\theta \sin\theta - \cos\theta), \tag{53}$$

which is biharmonic with a singularity at the origin, and can be shown to represent a concentrated force of magnitude -1, acting at the origin in the direction of the reference line ($\theta = 0$). It gives

$$\left.\begin{aligned}\frac{\partial\chi}{\partial x} &= -\frac{1}{2\pi}(2-\cos 2\theta),\\ \frac{\partial\chi}{\partial y} &= \frac{1}{2\pi}(2\theta+\sin 2\theta),\end{aligned}\right\} \tag{54}$$

when this reference line is directed along the x-axis.

We shall utilize this solution in our treatment of case (b), but in case (a), to illustrate the power of relaxation methods, we shall take the boundary conditions (51) as they stand. Hereafter we shall suppress the dashes attached to x, y, χ, which (with a) will accordingly stand for purely numerical quantities.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

Case (*a*).

39. In accordance with § 18 we attack this problem on two nets, independently, of the coarsest mesh ($a = 0{\cdot}4$), with the aim of taking into account at an early stage as much detail as possible of the specified edge tractions. Of the two interlacing nets in figure 22, the first (shown in full lines) has four nodal points in the quadrant for which residual forces have to be liquidated by a use of the standard 'pattern', the second (shown in broken lines) has two. These points are indicated by full and open circles respectively: the remainder are too near the boundary to permit relaxation.

On both nets liquidation is easy, and could be made complete if this were thought worth while: in fact it is not worth while until the 'selvedge values' (§ 22) have been amended, so on these coarse nets the work was done in short stages during which, alternately, interior points were relaxed in accordance with the standard pattern, 'selvedge points' in the light of rough plottings as described in §§ 20–2. Figure 22 records the accepted values, multiplied by 10^4 in order to obviate decimals.

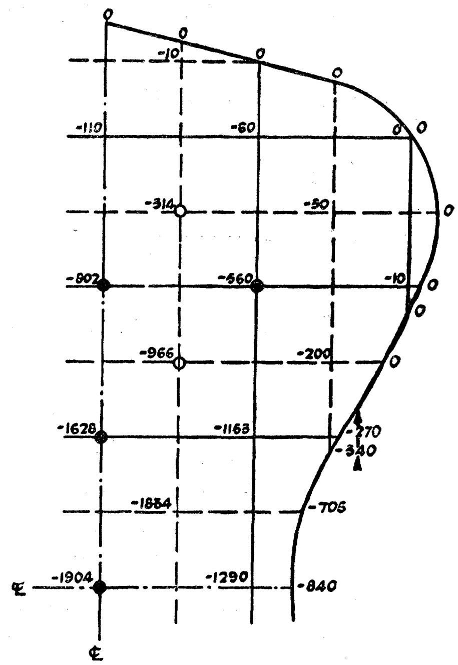


FIGURE 22

Figure 23 *a* records accepted values of χ at nodal points of a finer net comprising both of the interlacing nets of figure 22, and figure 23 *b* records values for a net which similarly interlaces the net of figure 23 *a*. In both, a (measured as before in terms of L) $= 0{\cdot}2$. Starting values were obtained from figure 22 by cross-plotting, and liquidation was effected with the aid of rough plottings as before. It substantially modified the starting values, as can be seen by comparing the values attached to common points of figures 22 and 23 *a*.

40. Cross-plottings of figures 23 *a* and *b* showed the two sets of values to be fairly consistent with one another, those of figure 23 *a* being, in general, slightly higher. Seeing no way of judging between them, we accepted both sets as starting values for the finer net ($a = 0{\cdot}1$) which is reproduced in figure 24, and we took mean values at other points of the new net. Thereby residual forces were obtained which, since positive values were interlaced with negative, could be liquidated without great labour. Figure 24 records the values finally accepted. Some adjustment was made to improve the approximation of the χ-gradients on the boundary near the 'waist'.

Case (*b*).

41. Our treatment of this case, in which the singularities entailed by concentrated forces were *eliminated initially* (§ 38), can be dismissed very briefly. Figure 25 records

accepted values of the three stress-components X_x, Y_y and X_y, which in relation to figure 24 would call for further computation with the use of the finite-difference approximations (35) and (36). It shows that the 'first uncertainty' of § 35 has no practical importance. (The numbers give X_x, Y_y and X_y as multiples of $100\,P/L$).

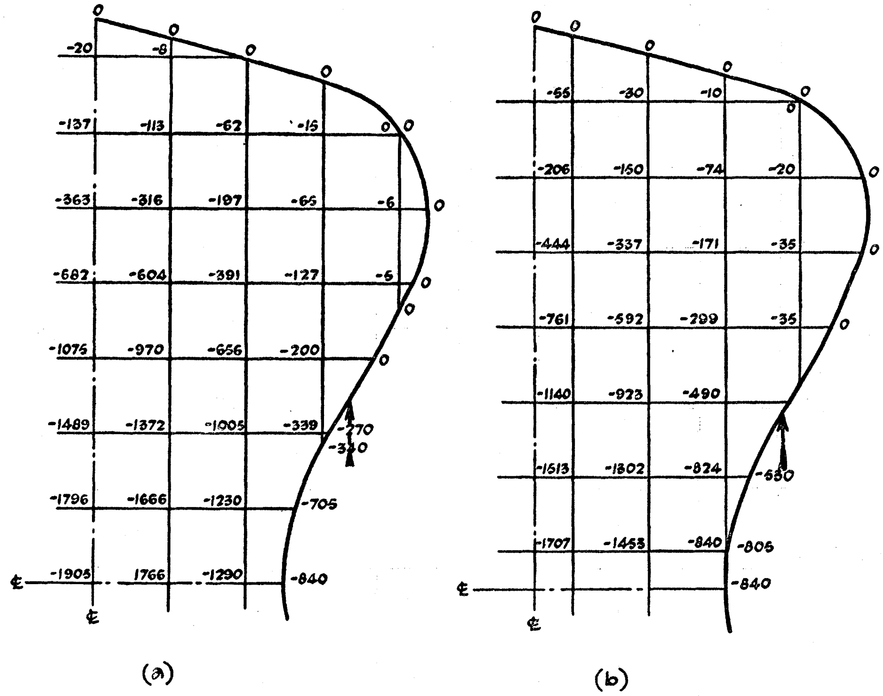


FIGURE 23

The system defined by (53) and (54) represents (§ 38) a force of magnitude -1 acting in the direction of the reference line ($\theta = 0$), so will hold in respect of the quadrant shown in figures 22–4 when the origin is taken at the point of loading and the x-direction from left to right. It gives boundary values of χ and of $\partial\chi/\partial x$ which are continuous, but values of $\partial\chi/\partial y$ which 'jump' at the point of loading as required by (52). By reflexion contributions can be computed for the concentrated forces at the other points of loading. Thereby a solution χ_0 (say) results which, with its differentials of any order, can be evaluated at every point. Its gradients do not satisfy (52); therefore, writing

$$\chi = \chi_0 + \chi_1$$

and substituting in (52), we have boundary values *without* discontinuity imposed on χ_1, $\partial\chi_1/\partial x$, $\partial\chi_1/\partial y$, and the methods of §§ 39–40 can be employed to determine χ_1.

More labour is entailed by this treatment than was expended on case (a). It is justified if, but only if, accurate values are wanted of the stresses close to the points of loading. In them χ_0 predominates, and the second differentials of χ_0 have exact analytical expressions: consequently a reasonably close approximation to χ_1 will yield results of more than proportionate over-all accuracy.

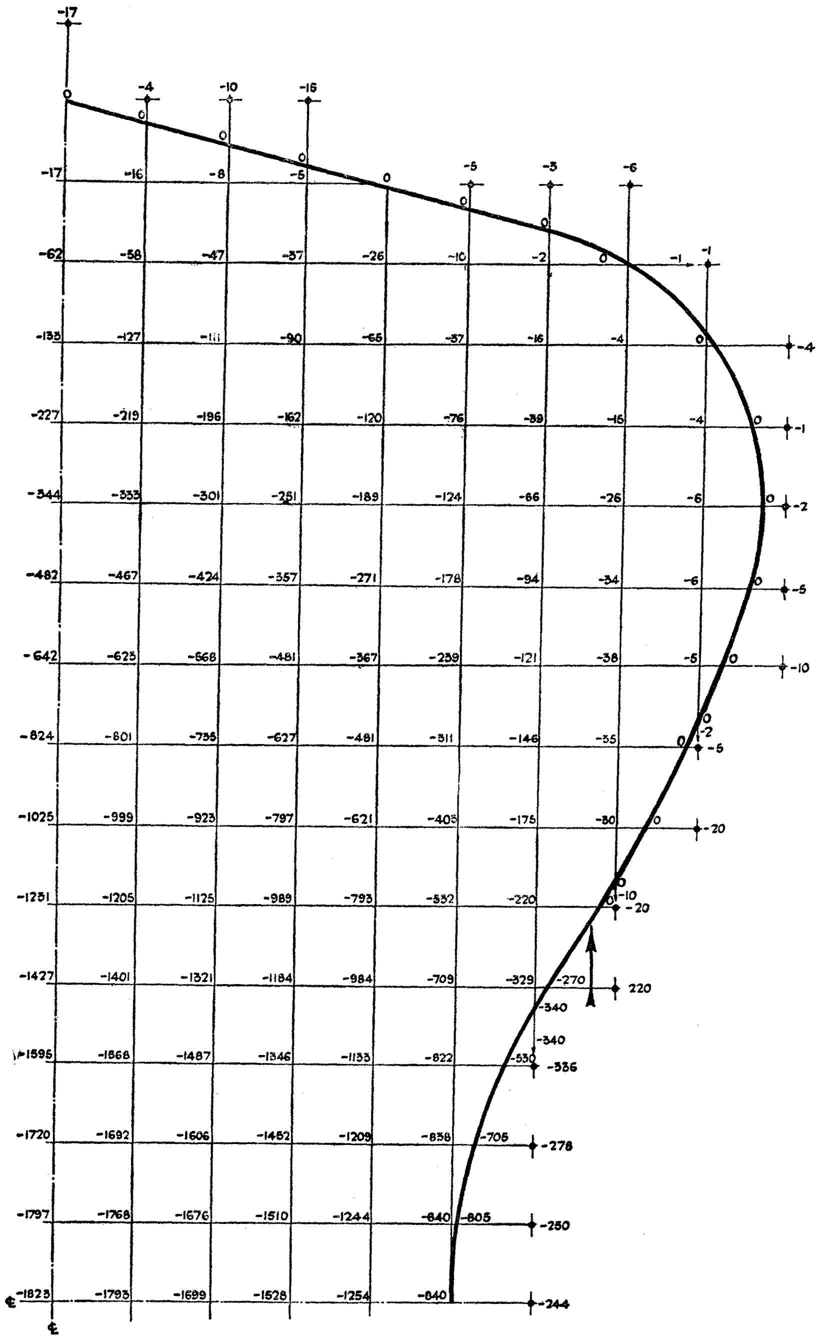


FIGURE 24. Accepted values of χ ($\times 10^4$) for case (*a*), § 36.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

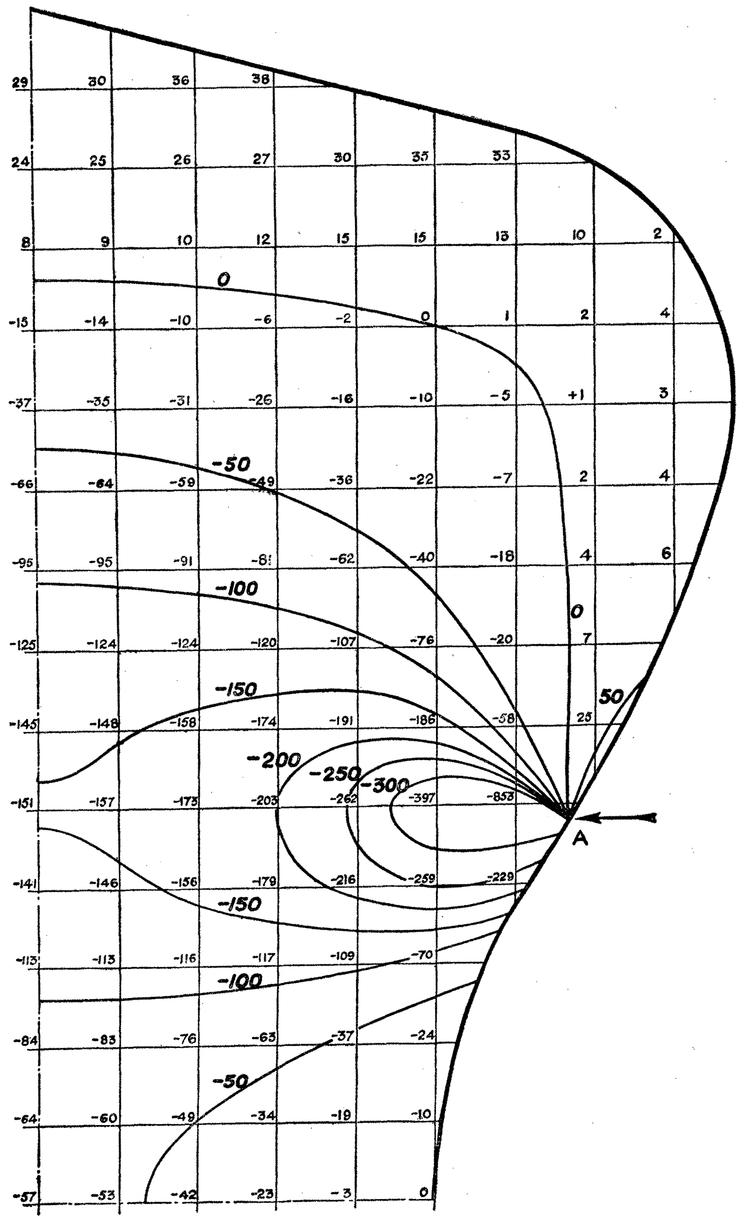


FIGURE 25*a*. Contours of X_x for case (*b*), § 36.

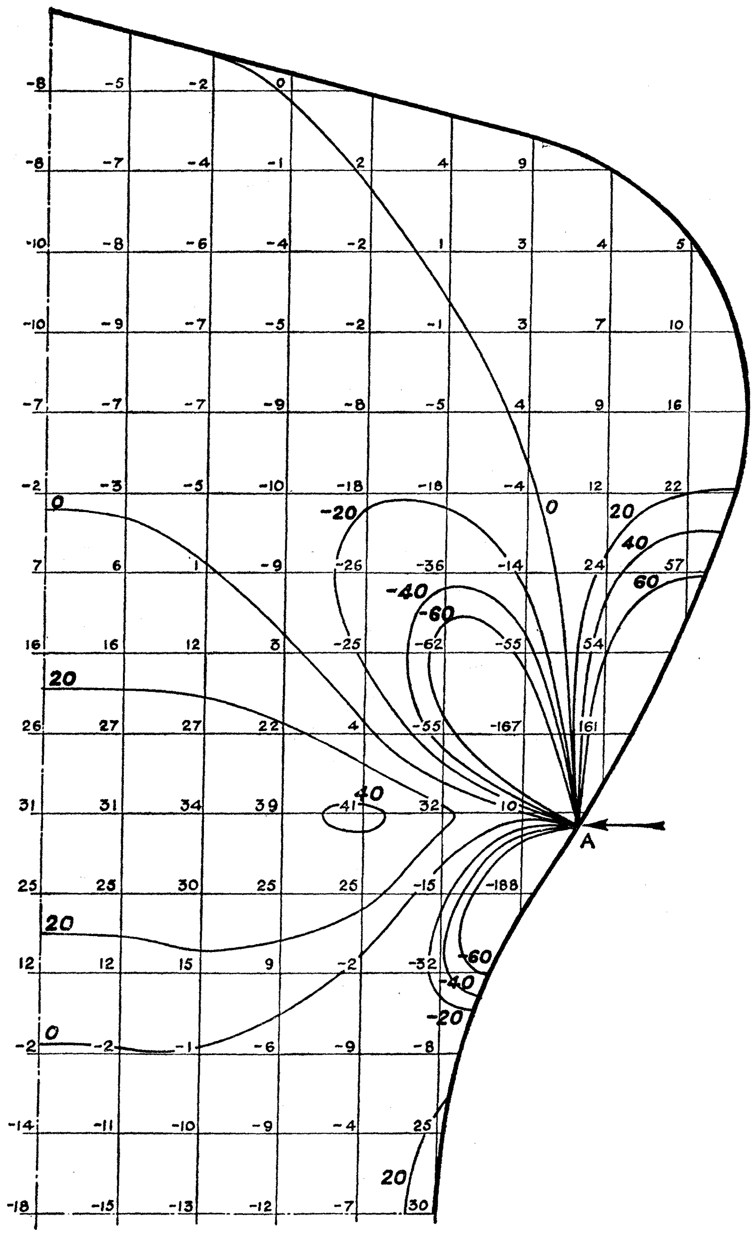


FIGURE 25*b*. Contours of Y_y for case (*b*), § 36.

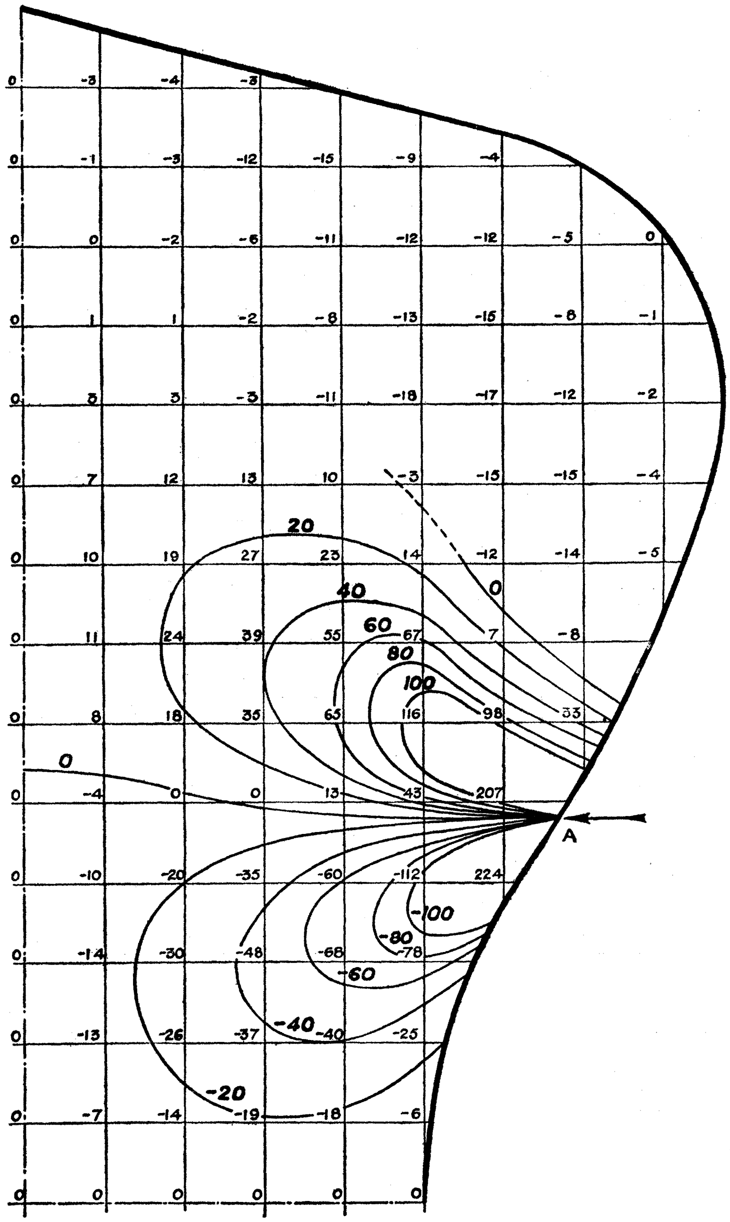


FIGURE 25c. Contours of X_y for case (b), § 36.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

IV. RÉSUMÉ

42. We have brought within the range of relaxation methods four important problems taken from the Theory of Elasticity, and we have solved (approximately) one example of each. But the technique developed in this paper has much wider application (cf. § 2). The main reason for our concentration on elastic problems (apart from their intrinsic interest and importance) is that in some of these the boundary conditions are peculiar and specially difficult.

For none of the four examples treated is an exact solution available for comparison, consequently no definite pronouncement can be made regarding accuracy.* All that can be claimed for our solutions is that a change in the last digit of any recorded value of w or χ would, *within the approximation of our finite-difference equations*, entail 'residual forces' greater than they have left unliquidated. Regarding the accuracy of the finite-difference equations it is not possible to generalize: we know the order in a of the quantities which they neglect, but we do not know their other factors, which are partial differentials of w or χ, so our estimates of resultant accuracy must be based mainly on intuition.

43. Figure 25 only excepted, we have left our solutions in the form of recorded values of w or χ at nodal points of a chosen net, holding that graphical presentation by means of contours is less accurate and no more convenient. Using (35) and (36), it is easy to deduce the curvatures and twist from w or X_x, Y_y, X_y from χ, and then the usual formulae can be used to deduce (if required) the principal curvatures or stresses. It does not appear that there is any generally accepted way of representing states of complex stress, although in photo-elastic studies it has become customary to give diagrams in which lines show at every point the directions of the principal stresses. Where three quantities can vary independently, three quantities are clearly necessary to a complete specification, and X_x, Y_y, X_y will do as well as any other three. Figure 25 presents these in the form of contours for case (*b*) of § 35.

In the nature of the case, second derivatives of a quantity determined by approximate methods are known with less certainty than the quantity itself; so whereas the accuracy of our recorded χ-values is greater than contour-plotting could have reproduced, this is not always true of the derived stress-values. None the less we believe that values obtained in this way from computed χ-values have an accuracy far beyond what is attainable by double differentiation of curves obtained by systematic sketching; and although the photo-elastic method starts with the advantage of dealing throughout with *stresses* (i.e. with second differentials of χ), we understand that there too 'smoothing' is usually practised.

* Love (1928 *a, b*) and Hencky (1913) have given analytical discussions of the square plate bent by uniform pressure, with some numerical results. For the central deflexion Hencky, using the Rayleigh-Ritz method, obtained a value $w = 1266$ (in our notation). Love gave 1672, 1344 and 1297 as successive approximations to the numerical coefficient, adding: 'I did no more arithmetic'.

44. There can be no question of the value of the photo-elastic method in presenting at once, and completely, a qualitative picture of the wanted stress-system. But quantitative results, as we believe, can be obtained with greater accuracy, and probably with less trouble, by the methods given in this paper, which are moreover applicable to problems not yet brought within the range of photo-elastic technique. Coker and Filon (1931, § 4.39) emphasize the relative simplicity of 'stress boundary conditions' and assert that '...little progress has been made hitherto in the study, by photo-elastic methods, of problems involving either conditions of displacement, or mixed conditions, at the boundary'.

45. It will be realized that in Section III, while we have explained the main steps in our solutions, space has not permitted a complete description of various devices which have been employed. As in all work of this kind, facility comes with experience, which suggests devices appropriate to each particular problem. The essence of Relaxation Methods (cf. Southwell 1940, §§ 17–20 and 266) is the tentative nature of its approach: *any* device may be employed to find a trial solution, since the subsequent treatment will take account of errors; and in particular (cf. § 21) rough plottings can be used in parallel with numerical computations, even when these aim at much higher accuracy.

Again, an apparently complete liquidation of residuals may prove to be incomplete when advance is made to a finer net. In such circumstances (cf. § 34) a multiplying factor may be used to ensure that the residuals, on the finer net, have *a zero total*: then positive and negative residuals will be equally common, and relatively little further relaxation will be required to liquidate them.

Finally, advantage can often be taken of the fact that (since the governing equation is linear) solutions can be superposed. As remarked in § 1, biharmonic analysis would be (relatively) simple if the boundary conditions permitted us to determine $\nabla^2 w$ as a first stage in the solution, because the deduction of w can be effected without difficulty, once $\nabla^2 w$ is known, by the methods of Part III. Similarly, if we have $\nabla^2 w$ approximately (from a solution effected by the methods of this paper), then we can without difficulty deduce a solution *w which satisfies very closely both the governing equation and either one of the two boundary conditions*. The other boundary condition will not (in general) be satisfied exactly, but it will be satisfied approximately; so the supplementary function required to complete our solution will be a biharmonic function which is everywhere small, therefore need not be determined with very high accuracy.

Conclusion

The methods of this paper can, as we believe, be applied to any problem of the kinds which it considers, and will yield results of more than sufficient accuracy for engineering purposes. They are laborious, but not more laborious (for comparable accuracy) than the photo-elastic technique which appears, at present, to be the sole alternative. It was to be expected that they would be considerably more laborious than the methods proposed for potential problems and the like in Part III of this series.

We acknowledge with gratitude help received from Messrs J. R. Green and F. S. Shaw, also from the Secretary and Staff of the Aeronautical Research Committee, in the preparation of ink drawings suitable for reproduction.

References

Christopherson, D. G. and Southwell, R. V. 1938 *Proc. Roy. Soc.* A, **168**, 317–350.
Coker, E. G. and Filon, L. N. G. 1931 *A treatise on photo-elasticity*. Camb. Univ. Press.
Gandy, R. W. G. and Southwell, R. V. 1940 *Phil. Trans.* A, **238**, 453–475.
Hencky, H. 1913 'Der Spannungszustand in rechteckigen Platten' (Diss.) Darmstadt: published by R. Oldenbourg, München u. Berlin.
Love, A. E. H. 1927 *A treatise on the mathematical theory of elasticity*, 4th ed. Camb. Univ. Press.
Love, A. E. H. 1928*a* *Proc. Lond. Math. Soc.* (ser. 2), **29**, 189–242.
Love, A. E. H. 1928*b* *Proc. Roy. Soc.* A, **118**, 427–440.
Michell, J. H. 1900 *Proc. Lond. Math. Soc.* **32**, 35–61.
Rayleigh, Lord 1926 *The theory of sound*, 2nd ed. Macmillan and Co.
Southwell, R. V. 1940 *Relaxation methods in engineering science*. Oxford Univ. Press.

MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES
THE ROYAL SOCIETY A
PHILOSOPHICAL TRANSACTIONS OF